FINANCIAL AND REAL OPTIONS THEORY AND LATTICE TECHNIQUES

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Abstract

The purpose of this dissertation is to propose a simple and general approach to solving real options problems. Most real options work to this point has been focused on complicated and specialized models. The focus here is to make real options modeling more robust and accessible. Although real options is the focus of this dissertation, many of the contributions are useful for financial modeling as well.

The main application areas are options on *observables*, which are general variables whose value can be observed. Observables can be classified as non-assets, market assets, or private assets. *Non-assets* are variables that are incapable of being traded but whose outcome is directly observable. Examples of non-assets include a company's market share, the total world demand for a product, a site's number of internet hits, and a competitor's decisions. Non-assets with no quantitative value can be given arbitrary values (e.g., a loan is either approved(1) or denied(0)). *Market assets* are items of monetary value that have a well-known price and are freely traded in an efficient marketplace like The New York Stock Exchange or NASDAQ. They could be stocks, bonds, or commodities; or, they could be derivatives of any of those. *Private assets* are items that either are not traded or are infrequently traded so that their current market value cannot be directly observed. We will estimate or compute their monetary value in this dissertation. The value of private assets may depend on the

value of other private assets, of non-assets, and of market assets. Examples of private assets include private companies, projects, and intangible assets like a brand name.

The contributions of this dissertation supplement the theory and improve the lattice techniques used to value real and financial options. More specifically, the contributions supplement the theory by handling non-asset underlying variables, by including learning effects, and by finding risk-neutral probabilities with new sets of information. The contributions also improve the lattice techniques by introducing and testing new lattices that are accurate, robust, simple, and able to handle both multiple underlying variables and learning effects.

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Contents

List of Tables

List of Figures

Chapter 1

Introduction

The purpose of this dissertation is to propose a simple and general approach to solving real options problems. Most real options work to this point has been focused on complicated and specialized models. The focus here is to make real options modeling more robust and accessible. Although real options is the focus of this dissertation, many of the contributions are useful for financial modeling as well.

This dissertation is intended to be read by academics but can be useful for business consultants, analysts, corporate decision makers, and others who are interested in building models to sharpen their intuition and make better decisions and valuations. It can also be used by traders, bankers, hedgers, investors, and others who are interested in accurately pricing market derivatives.

Real options is a theory that values non-financial investments and gives strategies for managing real assets (e.g., when to contract or expand, move or stop, build or tear down, speed up or slow down). For examples of real options problems, see Amram and Kulatilaka (1999), Trigeorgis (1995), and Dixit and Pindyck (1994); for a comprehensive review, see Trigeorgis (1993a) and Trigeorgis (1993b).

The main application areas are options on *observables*, which are general variables whose value can be observed. Observables can be classified as non-assets, market assets, or private assets. *Non-assets* are variables that are incapable of being traded but whose outcome is directly observable. Examples of non-assets include a company's market share, the total world demand for a product, a site's number of internet hits, and a competitor's decisions. Non-assets with no quantitative value can be given arbitrary values (e.g., a loan is either approved(1) or denied(0)). *Market assets* are items of monetary value that have a well-known price and are freely traded in an efficient marketplace like The New York Stock Exchange or NASDAQ. They could be stocks, bonds, or commodities; or, they could be derivatives of any of those. *Private assets* are items that either are not traded or are infrequently traded so that their current market value cannot be directly observed. We will estimate or compute their monetary value in this dissertation. The value of private assets may depend on the value of other private assets, of non-assets, and of market assets. Examples of private assets include private companies, projects, and intangible assets like a brand name.

Building real options models improves intuition, and the resulting strategies and valuations closely match that intuition. But the models are not robust enough to handle the variable growth rates and volatilities that occur outside the market. For example, growth rates of underlying observables may be expected to be exceptionally high for the next year or so but then drop off dramatically. In addition, we do not know how to handle options on non-assets.

The numerical techniques have limited usefulness as well. Multiple lattices are complex and Monte Carlo Simulation must have the strategy known in advance. Unfortunately, most projects contain several underlying observables (e.g., fixed costs, market share, and market size), and the optimal strategy is almost never known in advance.

To further complicate matters, more detailed forecasting is required because observables do not have implied volatilities and because the expected growth rate of non-assets will not necessarily be the same as that of similar-CAPM-Beta market assets. Growth rates, volatilities, and other parameters that most business experts are not comfortable with must be estimated. In addition, the optimal strategies and option valuations are usually sensitive to these parameters. To aid these expert forecasts, we will study historical data like past stock behavior. In many cases, however, data are difficult to obtain, and there are not enough to ensure a good forecast. Therefore, the judgment of the expert is paramount.

Most experts understand that their predictions are not perfect, and that if future results are not consistent with their predictions, they will alter those predictions. In other words, an expert cannot say with certainty what an expected growth rate or volatility will be, but instead can offer a probability distribution of these parameters. Then, as future results unfold, the expert becomes more certain of the appropriate parameters.

1.1 Contributions

The contributions of this dissertation supplement the theory and improve the lattice techniques used to value real and financial options. More specifically, the contributions supplement the theory by handling non-asset underlying variables, by including learning effects, and by finding risk-neutral probabilities with new sets of information. The contributions also improve the lattice techniques by introducing and testing new lattices that are accurate, robust, simple, and able to handle both multiple underlying variables and learning effects.

Contribution 1 (Non-Assets). We are able to include non-assets that are correlated with the Market portfolio in discrete or continuous time into our models. We construct a pricing theory that writes risk-neutral pricing solutions simply in both discrete (see sec. 2.1) and continuous time (see sec. 3.1 and sec. 3.3) for all observables (including non-assets). In discrete-time models, we can use non-assets (including those with no known value today) to find the risk-neutral probabilities (see sec. 2.4.2). In continuous-time models, we may need to transform underlying variables so that they follow geometric Brownian motion.

Contribution 2 (Layered Models). If we have an observable that has a continuous distribution or more outcomes than our model has states, we are able to find the risk-neutral expectations of the observable conditional on the state of the model (see sec. 2.2). Using information from these observables, we can find the risk-neutral probabilities (see sec. 2.4.2).

Contribution 3 (Closely-Correlated Assets). We can find the risk-neutral expectation of an observable by using information about a closely-correlated asset to the observable instead of the Market portfolio (see sec. 2.3). This method can add accuracy to our estimations of the risk-neutral probabilities (see sec. 2.4.2). We can also find the risk-neutral growth rate of an observable by using information about a closely-correlated asset (see sec. 3.2).

Contribution 4 (Market Portfolio Method). We add flexibility by introducing a new method that allows us to find the risk-neutral probabilities with just information about the Market portfolio (see sec. 2.4.1).

Contribution 5 (Learning). We find optimal strategies and valuations on models that include learning about the distribution of recurring event probabilities (see sec. 2.5), about the distribution of growth rates and volatilities of a continuous-time observable when we only observe its value periodically (see sec. 3.4), and about more general variables like the time to complete a project (see sec. 3.5). We assume that the distribution of the growth rates, volatilities, and recurring event probabilities are constant over time and use Bayesian Theory to update our beliefs about these parameters. For example, we make a prior distribution from expert opinion and historical data and then update that distribution as we observe data.

Contribution 6 (Lattice Tests and Comparisons). We test lattices over a comprehensive real options region (see sec. 4.4), and we compare the lattices by their simplicity, accuracy, and robustness (see sec. 4.1). The testing shows that the Cox, Ross, and Rubinstein(CRR) Lattice is inaccurate and that the best absolute expected growth rate to use with the CRR Lattice is 5%; the comparisons provide a good argument for the use of the binomial log-transform lattice(BI) (an approximation of risk-neutral geometric Brownian motion) and the trinomial log-transform lattice(TRI) (see sec. 4.2.4). The BI and TRI are always stable, have superior accuracy, and have unparalleled simplicity even with several underlying variables (see sec. 4.2.1). The TRI is used when we need greater range than the BI.

Contribution 7 (New Single Lattices). We construct new lattice types by approximating risk-neutral pricing solutions with moment-matching methods. Several of these lattice types prove to have binomial approximations that are significantly superior to the CRR Model. We make lattices by approximating four different models: the Log-Transform Model; the Multiplicative Model (an altered version of the

Log-Transform Model); the Alternative Log-Transform Model (similar to the Log-Transform Model but the risk-neutral expected growth rate is matched); and the Lognormal Model (same model that Tian (1993) approximated) (see sec. 4.3). We approximate the Log-Transform Model by using three different forms of moment-matching: pure (same as Omberg (1988)); equal-spacing (similar to Omberg (1988)'s equal-spacing lattices except that we set the spacing so that the first unmatched moment is as close as possible to being matched) (see sec. 4.2.2); and equal-spacing and equal-probability (see sec. 4.2.3).

Contribution 8 (New Multiple Lattices). We construct new, simple multiple lattices. By putting two BI lattices together, we create a simple double lattice that is always stable (see sec. 5.1.1). By putting three or four BI lattices together, we create effective multiple lattices that are simple to understand (see sec. 5.1.2).

Contribution 9 (Flexible Double Lattices). We make new double lattices out of combinations of the BI and trinomial lattices. Depending on the importance of each variable, the correlation between variables, and the needed range, we tailor the lattice for the particular problem and achieve a higher degree of accuracy (see sec. 5.1.3). Although these lattices are not as simple as others, they allow the flexibility to increase accuracy when one variable is more important than the other or when a larger than normal range is needed.

Contribution 10 (Unconditionally Stable Multiple Lattices). We build new multiple lattices that are unconditionally stable and consistently accurate for even a large number of underlying variables. We make Ekvall (1996)'s transformation and then form a lattice out of combinations of BI and trinomial lattices (see sec. 5.2 and sec. 5.4). Alternatively, we build a factor model and form a multiple lattice out of the factors (see sec. 5.3).

1.2 Future Research

With the contributions of this dissertation, learning effects are modeled, an observable with nearly any distribution is handled just as efficiently as a market asset, and we have simple, effective, lattice techniques. There are still several areas of research that could improve real options theory, however. In particular, future research should attempt to:

Remove Assumptions. The problems we attempt to model are complex, and many of our assumptions are unrealistic. We assume that we can hedge or sell unwanted risks; but hedging and selling are difficult, inefficient, and complex processes. In addition, observation and transaction costs are nontrivial.

Determine appropriate uses for real options models. Real options modeling is directly applicable to projects in large companies in three ways. The first consists of very small projects that are independent of, and have little affect on, surrounding areas of the business. The second is when used to produce a large overview of the entire business, considering restructuring costs, bankruptcy, synergies, and other factors that lead to risk-aversion. The third requires an overseer who decides how to break up, merge, buy, and sell divisions that each make decisions independently. Within a private company, we must be careful when considering using real options models because the owners generally prefer that the company act in a risk-averse manner.

Find ways of modeling non-specific distant opportunities. Kasanen (1993) suggests that corporations should seek investments that could spawn other opportunities.

Continue improving the lattice techniques. Although several lattice types are presented in this dissertation, there are many others that can be built and tested. More specifically, research should focus on testing multiple lattices and on discovering particularly effective lattices to use for specific classes or types of options.

Find lattice techniques for more general strategies. Lattice methods for controllable underlying variables and path-dependent options should be researched.

1.3 Organization

Here is the organization of the remaining chapters:

Chapter 2 explores discrete-time models. It shows how to find the risk-neutral expectations of observables, even if we do not know the current value of the observable, and even if the observable has continuous, negative, or more possible outcomes than the model has states. To add flexibility in our method of finding the risk-neutral expectation of an observable, we present a method of finding this risk-neutral expectation with information about a closely-correlated asset instead of information about the Market portfolio. The risk-neutral probabilities are found with information or conditional information from the Market portfolio or observables. We present an example of pricing with stochastic discount rates and of an observable modeling a binary event. To model learning about the distribution of probabilities, we use a beta distribution in an example with two periods, an example with 104 periods (using a special lattice type), and an example with an infinite number of periods.

Chapter 3 explores continuous-time models. We see how to adjust the expected growth rate of observables so that the observables fit into our risk-neutral framework. To add flexibility in our method of finding the risk-neutral growth rate of an observable, we show a method of finding this risk-neutral growth rate with information about a closely-correlated asset instead of information about the Market portfolio. We model learning about the growth rate or volatility of an observable when we only observe its value periodically. We show how to build a lattice when we learn about the growth rate. Finally, we study an application of learning in phases and of valuing an acquisition target that will allow a larger company to develop a new product. If the company buys the target and begins to develop the product, the non-intuitive optimal strategy suggests that, in many cases, the company should abandon the project before completion.

Chapter 4 develops, compares, and tests lattice methods. Lattices built from various continuous-time models and approximation methods are compared by their stability, simplicity, and accuracy. We approximate the Log-Transform Model with three moment-matching methods: pure; equal-spacing; and equal-spacing and equal-probability. We then review the CRR Model and present three other risk-neutral continuous-time models, which we also approximate with moment-matching methods. Finally, we test the lattices on 320 American options with parameters covering a comprehensive real options region. To determine the accuracy of the lattices, we calculate statistics of errors and percentage errors.

Chapter 5 covers multiple lattices. We use the BI to construct multiple lattices like the Double Binomial Lattice, the Triple Binomial Lattice, and the Quadruple Binomial Lattice. We then construct other double lattices out of lattice types like the Log-Transform Trinomial. To easily form lattices that contain a large number of underlying variables, we construct new lattices from the transformation of Ekvall (1996) and from a factor model. Finally, we see an application of valuing a lease in continuous time that contains multiple non-assets.

Chapter 2

Discrete-Time Pricing

The Capital Asset Pricing Model (CAPM) shows a relationship between market assets. To build the CAPM, we define the *return*, R_S, of any market asset, S, over time period t to $t+1$ as $R_S = S(t+1)/S(t)$, where $S(t)$ is the value of S at time t. We assume the market is efficient (i.e., there are no transaction costs, all investors have the same opportunities and information, no single person or company can significantly affect the Market portfolio with its actions, and no arbitrage opportunity exists) and that investors are strictly variance averse. We also assume that there exists a *Zero-Coupon Treasury bond*, B, a special market asset that has a guaranteed single fixed payoff $B(t+1) = R_f B(t)$, where R_f is the *risk-free return*. The CAPM states that the expected return, $\mathrm{E}(R_S)$, of a market asset S must satisfy

$$\mathrm{E}(R_S) - R_f = \beta_{R_S,R_M}[\mathrm{E}(R_M) - R_f] \tag{2.1}$$

where

$$\beta_{R_S,R_M} = \frac{\text{cov}(R_S, R_M)}{\text{var}(R_M)} \tag{2.2}$$

is the *CAPM Beta* of S (which we also refer to as the *Beta of* R_S *and* R_M), and M is the Market portfolio. Notice that the expected return of any market asset is determined solely by the covariance of the return of the Market portfolio and the return of the market asset. We can add dividends and net convenience yield to the expected return with only trivial changes to the analysis (see, e.g., Hull (1993, p. 68)). The net convenience yield of private assets is usually called *rate-of-return shortfall.* See Paddock, Siegel, and Smith (1988); Brennan and Schwartz (1985); McDonald and Siegel (1985); and McDonald and Siegel (1984) for examples of rate-of-return shortfall.

To price in models with L states, suppose there exists normalized state prices $q_i = \psi_i/\psi_0$ (see, e.g., Luenberger (1998a, pp. 248-51) and Huang and Litzenberger (1988, p. 124)), where $\psi_0 = \sum_{i=1}^{L} \psi_i$. We see that ψ_0 must equal $1/R_f$ by noting that the return of a bond over time period 0 to 1 is the same in every state and

$$B(0) = \sum_{i=1}^{L} \psi_i B(1) = \psi_0 B(1) = \frac{1}{R_f} B(1).$$

The q_i's sum to one and each of them is greater than zero just like a set of probabilities. The price of any market asset S with price S_i in state i and price S_0 today is

$$S_0 = \sum_{i=1}^{L} \psi_i S_i = \psi_0 \sum_{i=1}^{L} q_i S_i = \frac{1}{R_f} \hat{\text{E}}(S), \tag{2.3}$$

where $\hat{\mathrm{E}}(S)$ is the expectation of S under these artificial, *risk-neutral*, probabilities. Rearranging (2.3), we see that

$$\hat{\mathrm{E}}(R_S) = R_f \tag{2.4}$$

for all market assets.

To solve for the risk-neutral probabilities, we need information about a number of market assets - the same number as the number of states (see, e.g., Duffie (1988, pp. 67-8) and Huang and Litzenberger (1988, p. 124)). We assume that markets are complete. Along with the no arbitrage assumption, this means that the risk-neutral probabilities exist and are unique.

2.1 Risk-Neutral Expectations

We assume that in the next period, there are L states of the model, and that an observable Z (e.g., world demand for microprocessors or a stock price) has value Z_0 today, expected value $\mathrm{E}(Z)$ in the next period, and value Z_i in state i in the next period. State i will occur with probability p_i, where $\sum_{i=1}^{L} p_i = 1$.

Theorem 2.1. *For an observable Z with $Z_i > 0$ for each state i and $Z_0 > 0$,*

$$\mathrm{E}(R_Z) - \hat{\mathrm{E}}(R_Z) = \beta_{R_Z,R_M}[\mathrm{E}(R_M) - R_f], \tag{2.5}$$

where β_{R_Z,R_M} is the CAPM Beta of Z, $\hat{\mathrm{E}}$ is risk-neutral expectation, R_M is the return of the Market portfolio, and R_f is the risk-free return.

Proof. Take a non-dividend paying asset S with $Z_i = S_i > 0$ for every state i and

$Z_0\Lambda_Z = S_0$, where Λ_Z is a constant (since markets are complete, we can always find an S such that $S_i = Z_i$ for every state i). We then know that $\beta_{R_Z,R_M} = \beta_{R_S,R_M}\Lambda_Z$, $\mathrm{E}(R_Z) = \mathrm{E}(R_S)\Lambda_Z$, and

$$\hat{\mathrm{E}}(R_Z) = \hat{\mathrm{E}}(R_S)\Lambda_Z = R_f\Lambda_Z.$$

Substitution of β_{R_Z,R_M}, $\mathrm{E}(R_Z)$, and $\hat{\mathrm{E}}(R_Z)$ into the CAPM (2.1) gives us

$$\frac{\mathrm{E}(R_Z)}{\Lambda_Z} - \frac{\hat{\mathrm{E}}(R_Z)}{\Lambda_Z} = \frac{\beta_{R_Z,R_M}}{\Lambda_Z}[\mathrm{E}(R_M) - R_f],$$

and multiplying both sides by Λ_Z gives us the desired result. □

Theorem 2.2. *For any observable Z,*

$$\mathrm{E}(Z) - \hat{\mathrm{E}}(Z) = \beta_{Z,R_M}[\mathrm{E}(R_M) - R_f]. \tag{2.6}$$

Proof. Multiply both sides of (2.5) by Z_0. To see that (2.6) holds for any Z, notice that (2.6) does not contain the value of Z_0 and is unchanged by the transformation, $Z \mapsto Z + C$, where C is a constant. □

Example 2.1 (European Call Option on an Observable). Price a European call option C that expires next period and that pays Z dollars if exercised, where Z is an observable with $Z_i > 0$ for every state i.

The price of C is

$$\begin{aligned} C_0 &= \frac{1}{R_f}\hat{\mathrm{E}}[\max(Z,0)] = \frac{1}{R_f}\hat{\mathrm{E}}(Z) \\ &= \frac{1}{R_f}\{\mathrm{E}(Z) - \beta_{Z,R_M}[\mathrm{E}(R_M) - R_f]\}. \end{aligned}$$

Note that if Z is an asset, the value of C_0 is simply Z_0.

2.2 Layered Models

Suppose we want to build a model with L states but we have an observable that is more conveniently or realistically modeled with more than L states (e.g., a lognormally-distributed stock price or Market portfolio), we can still use this observable to help find the risk-neutral probabilities of our L states. An observable Z that is initially modeled with more than L states has a risk-neutral expectation conditional on arriving in state i, $\hat{\mathrm{E}}(Z_i)$, that is found using the following corollary to theorem 2.2:

Corollary. *For any observable Z and any state i,*

$$\mathrm{E}(Z_i) - \hat{\mathrm{E}}(Z_i) = \beta_{Z_i,R_{M_i}}[\mathrm{E}(R_{M_i}) - R_f], \tag{2.7}$$

where $\mathrm{var}(Z_i)$ *is the conditional variance of Z,* $\mathrm{E}(Z_i)$ *is the conditional expected value of Z, and $\beta_{Z_i,R_{M_i}}$ is the conditional Beta of Z and the return of the Market portfolio.*

Theorem 2.3. *For any observable Z,*

$$\hat{\mathrm{E}}(Z) = \sum_{i=1}^{L}\left\{\mathrm{E}(Z_i) - \beta_{Z_i,R_{M_i}}[\mathrm{E}(R_{M_i}) - R_f]\right\} q_i, \tag{2.8}$$

where q_i is the risk-neutral probability of arriving in state i.

Proof.

$$\hat{\mathrm{E}}(Z) = \sum_{i=1}^{L} \hat{\mathrm{E}}(Z_i) q_i. \tag{2.9}$$

Substituting (2.7) into (2.9) gives us the desired result. □

2.3 Closely-Correlated Asset

In practice, it is often more difficult to accurately estimate the CAPM Beta, β_{R_Z,R_M}, of an observable Z and the parameters of the Market portfolio than to estimate the correlation between Z and a closely-correlated asset S and the parameters of S (see Luenberger (2000) for the accuracy advantages of these estimations). For example, imagine that Z is the number of computer processors sold in the world in one month and that S is the stock price of a major processor manufacturer.

Theorem 2.4. *For an observable Z with $Z_i > 0$ for each state i and $Z_0 > 0$, and an asset S,*

$$\mathrm{E}(R_Z) - \hat{\mathrm{E}}(R_Z) = \frac{\beta_{R_Z,R_M}}{\beta_{R_S,R_M}}[\mathrm{E}(R_S) - R_f]. \tag{2.10}$$

Proof. Substituting $\mathrm{E}(R_M) - R_f$ of (2.5) into the CAPM (2.1) gives us the desired result. □

By estimating the correlation between Z and S, $\rho_{Z,S}$, and by assuming that the rest of the uncertainty in Z is uncorrelated with M (i.e., $\rho_{Z,M} = \rho_{Z,S}\rho_{S,M}$) (see Luenberger (1999) for justification of this assumption), we get the following corollary to theorem 2.4:

Corollary. *For an observable Z with $Z_i > 0$ for each state i and $Z_0 > 0$, and an asset S where $\rho_{Z,M} = \rho_{Z,S}\rho_{S,M}$,*

$$\mathrm{E}(R_Z) - \hat{\mathrm{E}}(R_Z) = \beta_{R_Z,R_S}[\mathrm{E}(R_S) - R_f]. \tag{2.11}$$

2.4 Risk-Neutral Probabilities

We present two major methods for finding the risk-neutral probabilities: the Market Portfolio Method; and the Observable Method. The method we choose depends on the information available to us. The Market Portfolio Method needs the expected value of the Market portfolio conditional on arriving in each state and the probability of arriving in each state, p_i. The Observable Method needs the same number of linearly independent observables as states. The Market Portfolio Method can be particularly useful when there are a large number of states since we only need information about one observable regardless of the number of states. The Observable Method is useful when we cannot use the Market Portfolio Method or when we can more accurately estimate the risk-neutral probabilities by using observables with closely-correlated assets than by using the Market Portfolio Method.

2.4.1 Market Portfolio Method

Theorem 2.5. *For each state i,*

$$q_i = p_i \left\{ 1 + \frac{[\mathrm{E}(R_M) - \mathrm{E}(R_{M_i})][\mathrm{E}(R_M) - R_f]}{\mathrm{var}(R_M)} \right\}. \tag{2.12}$$

Proof. Take an asset S with value $S_i = 1$ in state i and $S_j = 0$ for all $j \neq i$. Then we see that

$$E(R_S) = \frac{E(S)}{S_0} = \frac{p_i}{S_0}, \tag{2.13a}$$

$$R_f = \frac{\hat{E}(S)}{S_0} = \frac{q_i}{S_0}, \tag{2.13b}$$

and

$$\beta_{R_S,R_M} = \frac{\operatorname{cov}(S, R_M)}{\operatorname{var}(R_M)S_0} = \frac{E(SR_M) - E(S)E(R_M)}{\operatorname{var}(R_M)S_0} = \frac{p_i[E(R_{M_i}) - E(R_M)]}{\operatorname{var}(R_M)S_0}. \tag{2.13c}$$

Substitution of (2.13a), (2.13b), and (2.13c) into the CAPM (2.1) and multiplying both sides by S_0 gives us the desired result. □

Example 2.2 (Risk-Neutral Probabilities Using the Market Portfolio). In a four-state model, the Market portfolio has value 3 today and expected value 5 in state one, 4 in state two, 3 in state three, and 2 in state four. The probability of arriving in state one is .4, in state two is .3, in state three is .2, and in state four is .1. Assume the variance of the Market portfolio is 38 and the risk-free return is 1.05. Find the risk-neutral probabilities, q_i.

We find that the expectation of the Market portfolio is 4, and then use (2.12) to find $q_1 = .391$, $q_2 = .3$, $q_3 = .2045$, and $q_4 = .1045$.

2.4.2 Observable Method

Suppose we have a risk-free bond and at least $L - 1$ other observables, where L is the number of states in the model. To find the risk-neutral probabilities, q_i, we solve a set of L equations consisting of $\sum_{i=1}^{L} q_i = 1$ and $L-1$ of the ten equation types listed below (Q–1 to Q–10). If we know the exact value of Z in each state, we use equation types Q–1 to Q–5, which are derived from (2.4), (2.5), (2.6), and (2.11). If Z has more possible outcomes than the model has states, we use equation types Q–6 to Q–10, which are derived from (2.7), (2.8), and (2.11).

Q–1 For any asset S with R_{S_i} known for each state i:

$$\sum_{i=1}^{L} q_i R_{S_i} = R_f. \tag{2.14}$$

Q–2 For any observable Z with $Z_i > 0$ known for each state i and $Z_0 > 0$, and any asset S where $\rho_{Z,M} = \rho_{Z,S}\rho_{S,M}$:

$$\sum_{i=1}^{L} q_i R_{Z_i} = \mathrm{E}(R_Z) - \beta_{R_Z,R_S}[\mathrm{E}(R_S) - R_f].$$

Q–3 For any observable Z with Z_i known for each state i, and any asset S where $\rho_{Z,M} = \rho_{Z,S}\rho_{S,M}$:

$$\sum_{i=1}^{L} q_i Z_i = \mathrm{E}(Z) - \beta_{Z,R_S}[\mathrm{E}(R_S) - R_f].$$

Q–4 For any observable Z with $Z_i > 0$ known for each state i and $Z_0 > 0$:

$$\sum_{i=1}^{L} q_i R_{Z_i} = \mathrm{E}(R_Z) - \beta_{R_Z,R_M}[\mathrm{E}(R_M) - R_f].$$

Q–5 For any observable Z with Z_i known for each state i:

$$\sum_{i=1}^{L} q_i Z_i = \mathrm{E}(Z) - \beta_{Z,R_M}[\mathrm{E}(R_M) - R_f]. \tag{2.15}$$

Q–6 For any asset S:

$$\sum_{i=1}^{L} q_i \{\mathrm{E}(R_{S_i}) - \beta_{R_{S_i},R_{M_i}}[\mathrm{E}(R_{M_i}) - R_f]\} = R_f. \tag{2.16}$$

Q–7 For any observable Z with $Z_i > 0$ for each state i and $Z_0 > 0$, and any asset S where $\rho_{Z,M} = \rho_{Z,S}\rho_{S,M}$:

$$\sum_{i=1}^{L} q_i \{\mathrm{E}(R_{Z_i}) - \beta_{R_{Z_i},R_{S_i}}[\mathrm{E}(R_{S_i}) - R_f]\} = \mathrm{E}(R_Z) - \beta_{R_Z,R_S}[\mathrm{E}(R_S) - R_f].$$

Q–8 For any observable Z and asset S where $\rho_{Z,M} = \rho_{Z,S}\rho_{S,M}$:

$$\sum_{i=1}^{L} q_i \left\{\mathrm{E}(Z_i) - \beta_{Z_i,R_{S_i}}[\mathrm{E}(R_{S_i}) - R_f]\right\} = \mathrm{E}(Z) - \beta_{Z,R_S}[\mathrm{E}(R_S) - R_f].$$

Q–9 For any observable Z with $Z_i > 0$ for each state i and $Z_0 > 0$:

$$\sum_{i=1}^{L} q_i \{\mathrm{E}(R_{Z_i}) - \beta_{R_{Z_i},R_{M_i}}[\mathrm{E}(R_{M_i}) - R_f]\} = \mathrm{E}(R_Z) - \beta_{R_Z,R_M}[\mathrm{E}(R_M) - R_f].$$

Q–10 For any observable Z:

$$\sum_{i=1}^{L} q_i \left\{\mathrm{E}(Z_i) - \beta_{Z_i,R_{M_i}}[\mathrm{E}(R_{M_i}) - R_f]\right\} = \mathrm{E}(Z) - \beta_{Z,R_M}[\mathrm{E}(R_M) - R_f].$$

Example 2.3 (Simple Pricing). In a two-state model, an asset C pays 1 in state one and 2 in state two. A stock S has price 2 today and will be 3 in state one and 1 in state two. Assume the risk-free return, R_f, is 1.05. Find the current price of C.

First we solve for the risk-neutral probabilities, q_i, using Q–1(2.14):

$$\frac{3q_1 + q_2}{2} = R_f$$
$$q_1 + q_2 = 1.$$

Solving these two equations gives us $q_1 = .55$ and $q_2 = .45$. To find the price of C, we use (2.3):

$$C_0 = \frac{1}{R_f}\sum_{i=1}^{2} q_i C_i = \frac{11C_1 + .9C_2}{21} = \frac{29}{21}.$$

Example 2.4 (Using an Observable). In a two-state model, assume an observable Z will be 3 in state one, 1 in state two, and has a correlation with the return of the Market portfolio of .5. The variance of the return of the Market portfolio is 0.09 and the expectation is 1.15. Assume the probability of arriving in each state is 1/2 and the risk-free return is 1.05. Find the risk-neutral probabilities, q_i.

We first find that the variance of the value of Z in the next period is 1 so that we can use Q–5(2.15):

$$3q_1 + q_2 = \frac{11}{6}$$
$$q_1 + q_2 = 1.$$

Solving these two equations gives us $q_1 = 5/12$ and $q_2 = 7/12$.

Example 2.5 (Stochastic Discount Rates). In a two-state model, the current one-period risk-free return, R_f, is 1.05. At the end of the first period, the next one-period risk-free return will be 1.03 in state one or 1.07 in state two. The probability of arriving in each state is 1/2. The current price of a two-period zero-coupon Treasury bond B is 90. Find the risk-neutral probabilities, q_1 and q_2.

First note that the price of the bond in state one is $B_1 = 100/1.03$ and the price in state two is $B_2 = 100/1.07$. We solve for q_1 and q_2 by using Q–1(2.14):

$$B_1 q_1 + B_2 q_2 = B_0 R_f$$

$$q_1 + q_2 = 1;$$

and get $q_1 = .287$ and $q_2 = .713$.

Example 2.6 (Continuous-Variable Observable). In a two-state model layered within an infinite-state model, a stock price S is currently 2 and has a lognormal distribution in the next period with expectation in state one of 3 and expectation in state two of 1. The Market portfolio is also lognormally-distributed, and the covariance of the stock price with the Market portfolio is 1 in state one and 0.25 in state 2. The expected return of the Market portfolio is 1.2 in state one and 1.0 in state two. The variance of the return of the Market portfolio is 0.2 in state one and 0.06 in state two. Assume the risk-free return is 1.05. Find the risk-neutral probabilities, q_i.

We use Q–6(2.16):

$$\left[3 - \frac{1}{.2}(.15)\right] q_1 + \left[1 - \frac{.25}{.06}(-.05)\right] q_2 = S_0 R_f$$

$$q_1 + q_2 = 1.$$

We find $q_1 = .856$ and $q_2 = .144$.

Example 2.7 (Arbitrary Observable as an Event). Assume Z is a binary event (e.g., a possible loan approval) with probability p of a positive outcome. Arbitrarily assign the value 1 to a positive outcome and the value 0 to a negative outcome. The correlation between Z and the Market portfolio is .4, the variance of the return of the Market portfolio is 0.09, and the expectation of the return of the Market portfolio is 1.15. Assume the risk-free return is 1.05. Find the risk-neutral probability, q.

First find the variance of Z: $\mathrm{var}(Z) = p - p^2$; and then use Q–5(2.15) to find

$$q = p - \frac{.4\sqrt{p - p^2}}{3}.$$

2.5 Learning

If we do not know the parameters of the underlying observables in our model, we use a Bayesian framework to incorporate the aspects of learning that take place over time. We have multiple time periods $t = 0, \ldots, T$. Let $C_{\pi_m}(t)$ be the value of an asset at time t with current strategy π_m. The payoff of the asset at time t is a function G_t of our current strategy and N different observables with values $Z_1(t), \ldots, Z_N(t)$ at time t. We can switch from strategy π_m to strategy π_i with a fixed cost of $S(\pi_m, \pi_i)$

at time t. The value of the asset is

$$C_{\pi_m}(t) = \begin{cases} 0 & \text{for } t = T \\ \max_{\pi_i}\left(\frac{1}{R_f}\hat{\mathrm{E}}\{C_{\pi_m}(t+1) + G_t[\pi_i, Z_1(t), \ldots, Z_N(t)]\}\right. & \\ \qquad \left. -S_t(\pi_m, \pi_i)\right) & \text{for } t = 0, \ldots, T-1. \end{cases} \tag{2.17}$$

To model learning about the uncertainty surrounding the probability distribution of a recurring event with two states, we use a beta(α_1, α_2) distribution (see, e.g., Gelman, Carlin, Stern, and Rubin (1995, p. 481) and Law and Kelton (nd, pp. 338-9). The beta distribution has density function of the probability of event one occuring:

$$f(x) = \begin{cases} \frac{x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{\int_0^1 t^{\alpha_1-1}(1-t)^{\alpha_2-1}dt} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and expected probability of event one:

$$\mathrm{E}(p_1) = \frac{\alpha_1}{\alpha_1 + \alpha_2}.$$

Furthermore, it can be shown that if we take a sample z_t from $Z(t) \sim$ beta(α_1, α_2) and $z_t = 1$, then the distribution of $Z(t+1)$ is

$$\left[Z(t+1) \mid z_t = 1\right] \sim \text{beta}(\alpha_1 + 1, \alpha_2).$$

Table 2.1: Conditional Expected Values

Period 1	$\mathrm{E}[M_1(2)]$	$\mathrm{E}[M_2(2)]$	$\mathrm{E}[M_3(2)]$
State One	$[7M_1(1)]/6$	$M_1(1)$	$[5M_1(1)]/6$
State Two	$[3M_2(1)]/2$	$M_2(1)$	$[3M_2(1)]/4$
State Three	$[5M_3(1)]/3$	$M_3(1)$	$[5M_3(1)]/6$

For L states, we use the notation, beta$(\alpha_1, \ldots, \alpha_L)$. This distribution has expected probability of event j,

$$\mathrm{E}(p_j) = \frac{\alpha_j}{\sum_{i=1}^{L} \alpha_i}.$$

Example 2.8 (Risk-Neutral Learning). In a three-state, two-period model, assume the current value of the Market portfolio, M, is 4; the expected value of M conditional on arriving in state one is 6; the conditional expected value of M in state two is 4; and the conditional expected value of M in state three is 3. In addition, table 2.1 shows the expected values of the Market portfolio in period two conditional on the state of period one and period two. Find the risk-neutral probabilities of the first two periods, $q_i(0)$ and $q_i(1)$, given we have a beta$(1,1,1)$ distribution of the state probabilities, $p_i(0)$.

The probability of any of the three states occuring in period one is $1/3$. If state i occurs, then the probability of state i occuring again in period two is $1/2$ and the probability of any other state occuring in period two is $1/4$. By using (2.12), we find that the risk-neutral probabilities for period one are $q_1(0) = .286$, $q_2(0) = .343$, and $q_3(0) = .371$. The risk-neutral probabilities for period two are solved similarly and are shown in table 2.2.

Table 2.2: Risk-Neutral Probabilities

Period 1	$q_1(1)$	$q_2(1)$	$q_3(1)$
State One	.527	.246	.227
State Two	.232	.505	.263
State Three	.209	.256	.535

Example 2.9 (Two Years of Learning). Suppose a bank has the opportunity to offer a new type of \$1 million short-term loan that could either pay \$2 million or \$0 in one week. We believe that the probability of success is uniformly distributed between 0 and 1, is uncorrelated with the Market portfolio, and is constant over time. We have the same opportunity once per week for the next two years. The risk-free return over one week is 1.0047.

First, note that a Uniform distribution U(0,1) is equivalent to a beta(1,1) distribution. Let $C(\alpha_1, \alpha_2, t)$ represent the value of the option at time t with the belief that the probability of success has a beta(α_1,α_2) distribution. From (2.17),

$$C(\alpha_1,\alpha_2,t) = \begin{cases} 0 & \text{for } t = T \\ \max\Big\{\frac{\alpha_1}{R_f(\alpha_1+\alpha_2)}[\$2\text{M} + C(\alpha_1+1,\alpha_2,t+1)] & \\ \quad + \frac{\alpha_2}{R_f(\alpha_1+\alpha_2)}C(\alpha_1,\alpha_2+1,t+1) - \$1\text{M}, 0\Big\} & \text{for } t = 0,\ldots,T-1 \end{cases}$$

For this example, it is necessary to build a lattice with changing probabilities similar to that in figure 2.1. If we take a myopic view and consider only the value of the first loan, we do not give the loan since it has an expected value of 0. If we consider the effects of learning, however, we will take on a more aggressive strategy and find the

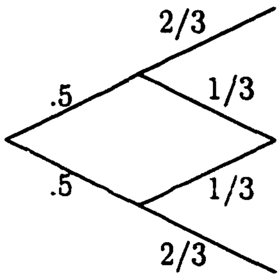


Figure 2.1: A uniform lattice.

value of the option to be \$23.02 million. The optimal strategy for the first twenty loans is shown in figure 2.2. Notice that we are willing to accept several unsuccessful outcomes before abandoning the new type of loan in hopes that we are just having bad luck.

To show further the effects of learning, we solve for the value of the option again, this time using a beta(50,50) distribution (see fig. 2.3), which also has $E(p_1) = .5$. The value is only \$1.81 million because of the reduced learning effect. Figure 2.4 shows the optimal strategy for the first twenty loans. We are willing to continue even longer because we believe that p_1 does not fall far from one-half.

Example 2.10 (Infinite Horizon Learning). Suppose that a bank has a division that offers only one particular type of loan. The loan either pays P_1 or P_2 in each period. Suppose also that we believe the probabilities of the payoffs have a beta(α_1, α_2) distribution, are uncorrelated with the Market portfolio, and are constant over time. The bank has the option to sell the division at any time for S. The risk-free return is R_f. We want to solve for the value of the division at time t, $C(\alpha_1, \alpha_2, t)$.

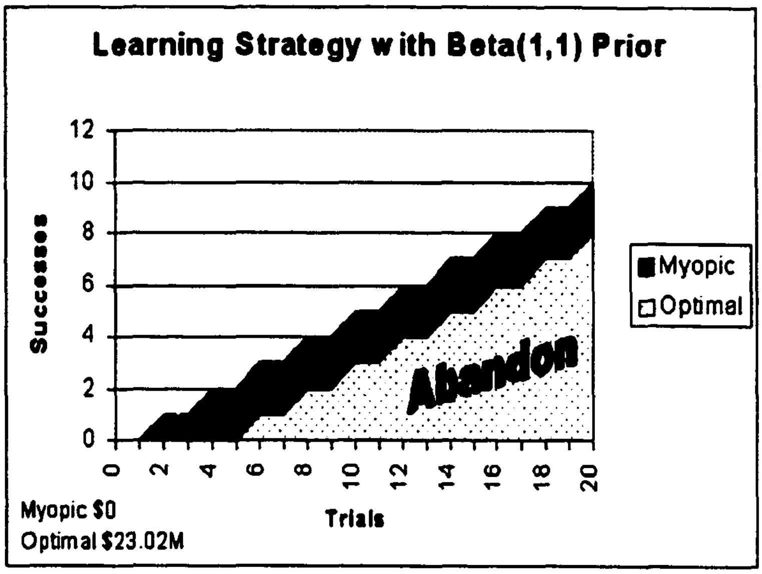


Figure 2.2: The beta(1,1) strategy.

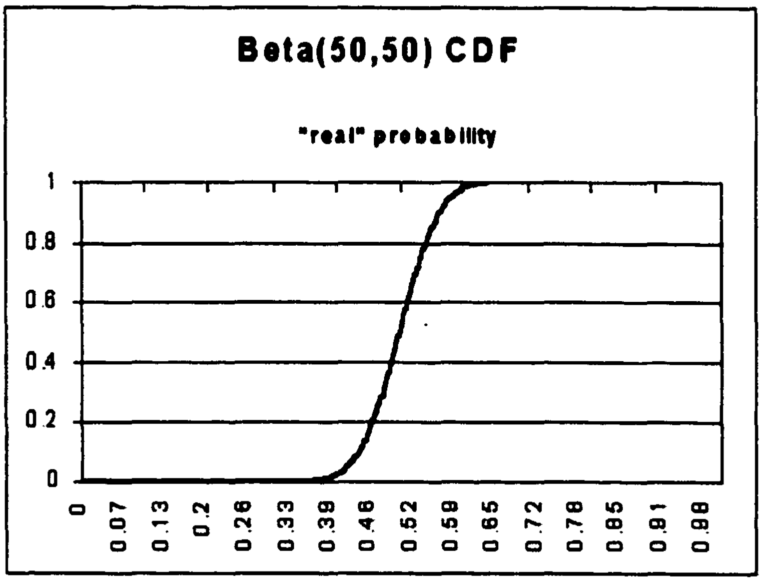


Figure 2.3: The beta(50,50) CDF.

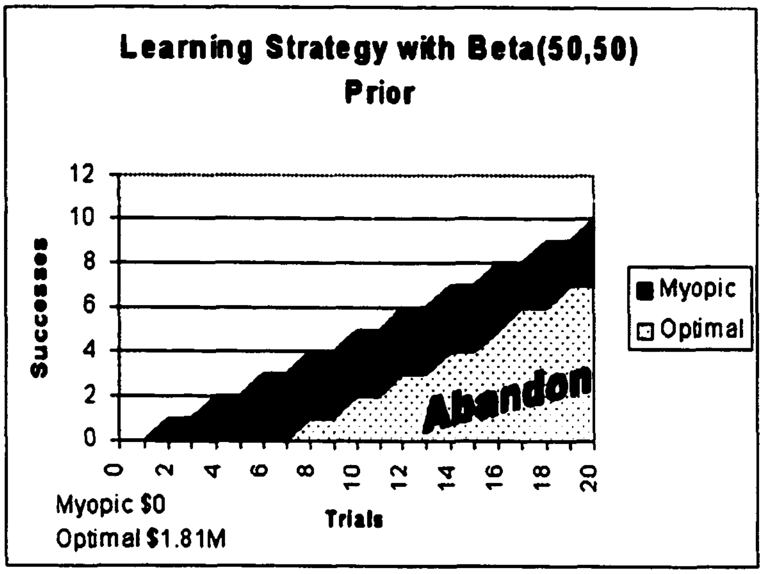


Figure 2.4: The beta(50,50) strategy.

From (2.17),

$$C(\alpha_1, \alpha_2, t) = \max\left\{\frac{\alpha_1}{AR_f}[P_1 + C(\alpha_1 + 1, \alpha_2, t+1)] \right.$$
$$\left. + \frac{\alpha_2}{AR_f}[P_2 + C(\alpha_1, \alpha_2 + 1, t+1)], S\right\},$$

where $A = \alpha_1 + \alpha_2$. For infinite horizon problems, as A approaches infinity, we know the real probabilities and write

$$C(\alpha_1, \alpha_2, t) = \max\left\{\frac{\alpha_1}{AR_f}[P_1 + C(\alpha_1, \alpha_2, t+1)] + \frac{\alpha_2}{AR_f}[P_2 + C(\alpha_1, \alpha_2, t+1)], S\right\},$$

which can then be rewritten

$$C(\alpha_1, \alpha_2, t) = \max\left[\frac{P_1\alpha_1 + P_2\alpha_2}{A(R_f - 1)}, S\right] \text{ for } R_f > 1 \text{ and } A >> 0.$$

To approximate the optimal stopping strategy and value of the option, start from a large A and many possible real probabilities and work backwards using dynamic programming.

Chapter 3

Continuous-Time Pricing

Financial Economic theory usually assumes market assets follow a *Markov process*, which is a special stochastic process in which only the present value of the variable is relevant for predicting the future. Markov processes are consistent with the weak form of market efficiency, which occurs because of rational competition. In reality, the prices of market assets do not move randomly, but instead move because of trades made by investors. To an outsider looking at the prices with no other information, however, the prices appear to move randomly. Empirically, the logarithms of stock prices closely follow a Normal price distribution. The most common model for these asset price processes is *geometric Brownian motion*, which is a lognormally-distributed, continuous-time, continuous-variable, Markov process (see, e.g., Oksendal (1995, pp. 9-12) and Duffie (1988, p. 231)). For a market asset S, this process takes the form

$$\mathrm{d}\ln(S) = \nu_S \mathrm{d}t + \sigma_S \mathrm{dB}_S, \tag{3.1}$$

or the equivalent form

$$\mathrm{d}S = \left(\nu_S + \frac{1}{2}\sigma_S^2\right) S\mathrm{d}t + \sigma_S S\mathrm{dB}_S, \tag{3.2}$$

where B_S is the Brownian motion of S, ν_S is the *expected growth rate* of S and σ_S is the *volatility* of S. We can add dividends and net convenience yield to the expected growth rate with only trivial changes to the analysis.

We assume that the Market portfolio follows the process

$$\mathrm{d}\ln(M) = \nu_M \mathrm{d}t + \sigma_M \mathrm{dB}_M.$$

This is consistent with both the log-optimal portfolio (see Luenberger (1998a, p. 432)) and the continuously-rebalanced optimal Markowitz portfolio (see, e.g., Luenberger (1998a, p. 433)). To find the log-optimal portfolio, we assume that investors only consider long-term performance; to find the Markowitz portfolio, we assume that investors are strictly variance averse. We also assume that the market is efficient (i.e., there are no transaction costs, all investors have the same opportunities and information, no single person or company can significantly affect the Market portfolio with its actions, and no arbitrage opportunity exists) and that there exists a Zero-Coupon Treasury bond B that follows the process:

$$\mathrm{d}\ln(B) = r_f \mathrm{d}t;$$

where r_f is the *risk-free rate*.

We assume that a risk-neutral Brownian motion (Brownian motion under the risk-neutral probability measure) can be formed for any market asset S:

$$\mathrm{d}\hat{\mathrm{B}}_S = \mathrm{dB}_S + \left(\frac{\nu_S + \frac{1}{2}\sigma_S^2 - r_f}{\sigma_S}\right)\mathrm{d}t \tag{3.3}$$

(see, e.g., Duffie (1996, p. 112)). Therefore, we can rewrite (3.1) as

$$\mathrm{d}\ln(S) = \nu_{S_Q}\mathrm{d}t + \sigma_S \mathrm{d}\hat{\mathrm{B}}, \tag{3.4}$$

where

$$\nu_{S_Q} = r_f - \frac{1}{2}\sigma_S^2$$

is the risk-neutral expected growth rate. Note that this risk-neutral expected growth rate satisfies (2.3).

To solve for the value of an American-style option of time length T, we break up the time length into n time steps of length $\Delta t = T/n$. Then, for example, we rewrite (3.4) as the discrete-time equation,

$$\Delta\ln(S) = \left(r_f - \frac{1}{2}\sigma_S^2\right)\Delta t + \sigma_S\sqrt{\Delta t}\Delta\hat{\mathrm{B}},$$

and approximate using lattice techniques (see ch. 4).

To solve for the value of a European-style option, we may only need to know the price of S at a few time points. Seen from time t, the price of S at time $t+1$ has the lognormal distribution:

$$S(t+1) \sim S(t)\mathrm{LN}(\nu_S\Delta t, \sigma_S^2\Delta t).$$

The mean of $S(t+1)$ is $S(t)\exp[\nu_S \Delta t + \sigma_S^2(\Delta t)/2]$. And, the risk-neutral distribution at time $t+1$ is

$$S_Q(t+1) \sim S(t)\mathrm{LN}\left(r_f \Delta t - \frac{1}{2}\sigma_S^2 \Delta t, \sigma_S^2 \Delta t\right).$$

To price a standard European call or put option in continuous time, we use a closed-form solution of the Black-Scholes option pricing equation (see, e.g., Luenberger (1998a, p. 351)). The Black-Scholes call option formula assumes a known and constant volatility of a lognormally-distributed underlying asset. If the price of the option is known, we are able to calculate this volatility. The Black-Scholes pricing equation also prices a limited number of other simple derivatives. For more information on financial theory, see Luenberger (1998a); Duffie (1996); and Huang and Litzenberger (1988).

Unless otherwise specified, we assume that the volatilities and interest rates in our models are constant over time. With only trivial changes to the analysis, these parameters can be made a function of time and proportional dividends can be included.

3.1 Risk-Neutral Expectations

Theorem 3.1. *For any observable Z that follows (3.1) (or can be transformed to follow (3.1)),*

$$\nu_Z - \nu_{Z_Q} = \beta_{Z,M}\left(\nu_M + \frac{1}{2}\sigma_M^2 - r_f\right), \tag{3.5}$$

where $\beta_{Z,M} = \rho_{Z,M}\sigma_Z/\sigma_M$ *and*

$$\rho_{Z,M} = \frac{\mathrm{E}(\mathrm{dB}_Z \mathrm{dB}_M)}{\mathrm{d}t}$$

is the correlation between Z *and the Market portfolio.*

Proof. First note that we can rewrite (3.1) as

$$\mathrm{d}\ln(Z) = \nu_Z \mathrm{d}t + \sigma_Z \rho_{Z,M} \mathrm{dB}_M + \sigma_Z \sqrt{1 - \rho_{Z,M}^2}\, \mathrm{dB}_I, \tag{3.6}$$

where B_M is the Brownian motion of the Market portfolio and B_I is a Brownian motion independent of B_M. Substituting $\mathrm{d}\hat{\mathrm{B}}_M$ from (3.3) for dB_M gives us the desired result. □

3.2 Closely-Correlated Asset

In practice, it is often more difficult to accurately estimate the parameters of the Market portfolio than to estimate the parameters of an asset S that is closely correlated with our observable Z. For example, imagine that Z is the market share of Intel in computer processors sold in the world and that S is the stock price of a major processor manufacturer. We can find the growth rate of Z with the parameters of S by estimating the correlation between Z and S, $\rho_{Z,S}$, and by assuming that the rest of the uncertainty in Z is uncorrelated with M (i.e., $\rho_{Z,M} = \rho_{Z,S}\rho_{S,M}$).

Theorem 3.2. *For any observable* Z *and asset* S *that follow (3.1) where* $\rho_{Z,M} =$

$\rho_{Z,S}\rho_{S,M}$,

$$\nu_Z - \nu_{Z_Q} = \beta_{Z,S}\left(\nu_S + \frac{1}{2}\sigma_S^2 - r_f\right). \tag{3.7}$$

Proof. First note that we can rewrite (3.1) as

$$\mathrm{d}\ln(Z) = \nu_Z \mathrm{d}t + \sigma_Z \rho_{Z,S} \mathrm{dB}_S + \sigma_Z \sqrt{1 - \rho_{Z,S}^2}\, \mathrm{dB}_I, \tag{3.8}$$

where B_S is the Brownian motion of the closely-correlated asset and B_I is a Brownian motion independent of B_S (and B_M). Substituting $\mathrm{d}\hat{\mathrm{B}}_S$ from (3.3) for dB_S gives us the desired result. □

3.3 Risk-Neutral Growth Rates

To find the risk-neutral expected growth rate of an observable Z, ν_{Z_Q}, we use the following three equation types (G–1 to G–3) derived from theorem 3.1 and 3.2.

G–1 For any asset S:

$$\nu_{S_Q} = r_f - \frac{1}{2}\sigma_S^2. \tag{3.9}$$

G–2 For any observable Z and asset S with $\rho_{Z,M} = \rho_{Z,S}\rho_{S,M}$:

$$\nu_{Z_Q} = \nu_Z - \beta_{Z,S}\left(\nu_S + \frac{1}{2}\sigma_S^2 - r_f\right). \tag{3.10}$$

G–3 For any observable Z:

$$\nu_{Z_Q} = \nu_Z - \beta_{Z,M}\left(\nu_M + \frac{1}{2}\sigma_M^2 - r_f\right). \tag{3.11}$$

Note that for the log-optimal portfolio, Luenberger (1998a, p. 436) shows that

$$\nu_M + \frac{1}{2}\sigma_M^2 - r_f = \sigma_M^2.$$

This simplifies our analysis since we only need to estimate the growth rate or volatility of the Market portfolio, instead of both. With the log-optimal portfolio, G–3(3.11) reduces to

$$\nu_{Z_Q} = \nu_Z - \beta_{Z,M}\sigma_M^2$$

or

$$\nu_{Z_Q} = \nu_Z - \rho_{Z,M}\sigma_Z\sqrt{2(\nu_M - r_f)}.$$

Example 3.1 (European Call Option). Price a European call option C on an asset S with strike price K that expires at time 1. Assume $S(1) \sim S(0)\mathrm{LN}(0,1)$ and the risk-free rate is 0.05.

The value of C can be written as

$$C_0 = \mathrm{e}^{-.05}\int_K^{\infty}[S_Q(1) - K]\mathrm{d}S_Q(1),$$

and from G–1(3.9), we calculate that

$$S_Q(1) \sim S(0)\mathrm{LN}(-0.451, 1).$$

Example 3.2 (Normally-Distributed Observable). Assume that an observable Z follows the process:

$$\mathrm{d}Z = \mathrm{dB}_Z;$$

that the correlation between e^Z and the Market portfolio, $\rho_{\mathrm{e}^Z,M}$, is .5; that the volatility of the Market portfolio is 0.3; that the expected growth rate of the Market portfolio is 0.14; and that the risk-free rate is 0.05. Find the risk-neutral process of Z.

We first notice that e^Z follows the process:

$$\mathrm{d}\ln(\mathrm{e}^Z) = \mathrm{dB}_Z.$$

Therefore, $\nu_{\mathrm{e}^Z} = 0$ and $\sigma^2_{\mathrm{e}^Z}(1) = 1$. We can then calculate $\nu_{\mathrm{e}^Z Q}$ using G–3(3.11):

$$\nu_{\mathrm{e}^Z Q} = \frac{-.5(1)}{.3}\left(.14 + \frac{.09}{2} - .05\right) = -.225;$$

and thus,

$$\mathrm{d}Z = -.225\mathrm{d}t + \mathrm{d}\hat{\mathrm{B}}_Z.$$

3.4 Learning

In this section, we explore the effects of an unknown expected growth rate or unknown volatility of an observable Z on the value of an option. If both are unknown, proper updating can be handled but not by a closed formula, so it is left to the interested reader (see, e.g., Gelman, Carlin, Stern, and Rubin (1995)). Throughout this section, we assume that Z is observed at discrete time intervals of length Δt and is uncorrelated with the Market portfolio.

3.4.1 Growth Rate

Assume that we do not know the expected growth rate of an observable Z, ν_Z, but instead believe that it is Normally distributed:

$$\nu_Z(t)\Delta t \sim \mathrm{N}[m(t)\Delta t, \tau^2(t)\Delta t].$$

After we observe a return z_{t+1} from a distribution

$$R_Z(t+1) \sim \mathrm{LN}[\nu_Z(t)\Delta t, \sigma_Z^2\Delta t] \sim \mathrm{LN}[m(t)\Delta t, \sigma_Z^2\Delta t + \tau^2(t)\Delta t], \tag{3.12}$$

the variance in our estimate becomes

$$\tau^2(t+1) = \frac{\sigma_Z^2\tau^2(t)}{\sigma_Z^2 + \tau^2(t)}. \tag{3.13}$$

and our estimate of the mean becomes

$$m(t+1)\Delta t = \frac{m(t)\sigma_Z^2(\Delta t)^2 + \tau^2(t)\Delta t \ln(z_{t+1})}{\sigma_Z^2\Delta t + \tau^2(t)\Delta t} \tag{3.14}$$

(see, e.g., Gelman et al. (1995, pp. 42-5)). In general, to update our estimate of the mean after n samples:

$$m(t+n)\Delta t = \frac{m(t)\sigma_Z^2(\Delta t)^2 + \tau^2(t)\Delta t \sum_{i=1}^{n} \ln(z_{t+i})}{\sigma_Z^2\Delta t + n\tau^2(t)\Delta t}. \tag{3.15}$$

Note that the learning effect is dependent on the initial uncertainty about the growth rate, the volatility, and the time between observations (which should at least be long enough that the continuous-time assumption is reasonable).

Example 3.3 (Learning About Growth Rate). Suppose we want to value an option C that pays Z dollars if exercised, where Z is an observable. The value of Z can be seen and the option can be exercised at time 1 or time 2. Also suppose the current value of Z is 1, the return of Z has distribution

$$R_Z(t) \sim \mathrm{LN}[\nu_Z(t), .2],$$

and we believe the expected growth rate has distribution

$$\nu_Z(t) \sim \mathrm{N}[m(t), \tau^2(t)],$$

where $m(0) = 0$ and $\tau^2(0) = 0.5$. Assume the risk-free rate is 0.05.

From (3.12), we can write the distribution of the observable as

$$Z(t+1) = Z(t)\mathrm{LN}[m(t), .2 + \tau^2(t)].$$

We calculate the distribution of the observable at time 1 as

$$Z(1) \sim \mathrm{LN}(0, .7).$$

Using (3.13) and (3.14), we update the variance of our estimate of the expected growth rate:

$$\tau(1) = \frac{\tau^2(0)\sigma_Z^2}{\tau^2(0) + \sigma_Z^2} = \frac{1}{7};$$

and the mean of our estimate of the expected growth rate:

$$m(1) = \frac{\sigma_Z^2 m(0) + \tau^2(0)\ln[Z(1)]}{\sigma_Z^2 + \tau^2(0)} = \frac{5}{7}\ln[Z(1)].$$

Then we can calculate the distribution of the observable at time 2:

$$Z(2) \sim Z(1)\mathrm{LN}\left[m(1), \frac{12}{35}\right].$$

We exercise at the end of year one if, and only if, the expected return of the observable is less than the risk-free return. That is, we exercise if

$$\begin{aligned} \hat{\mathrm{E}}[R_Z(2)] &= \exp\left\{m(1) + \frac{1}{2}[\sigma_Z^2 + \tau^2(1)]\right\} \\ &= \exp\left\{\frac{5}{7}\ln[Z(1)] + .0397\right\} < e^{.05}, \end{aligned}$$

which is equivalent to $Z(1) < 1.015$. The value of the call option is the sum of the payoffs if we exercise and if we do not exercise:

$$C_0 = \int_0^\infty \int_{1.015}^\infty Z(2)\mathrm{d}Z(1)\mathrm{d}Z(2) + \int_0^{1.015} Z(1)\mathrm{d}Z(1).$$

3.4.2 Volatility

Now assume that we know the expected growth rate but do not know the volatility. Instead, we believe that the square of the volatility has an inverse chi-squared distribution:

$$\sigma_Z^2(t)\Delta t \sim \text{Inv-}\chi^2[\kappa(t)\Delta t, \varsigma^2(t)\Delta t]$$

(chosen for its updating properties). The notation Inv-$\chi^2(\kappa, \varsigma)$ represents $\kappa\varsigma/\chi^2$ where κ is the number of degrees of freedom and ς is a scaling factor (Gelman et al. 1995). The density function of Inv-χ^2 can be found in Gelman et al. (1995, p. 474). Note that κ can be interpreted as the number of observations and that ς can be interpreted as the average squared deviation of those observations. The expected value of the squared volatility is

$$\mathrm{E}[\sigma_Z^2(t)] = \frac{\kappa(t)\varsigma^2(t)}{\kappa(t) - 2}.$$

After we observe a return z_{t+1} from a distribution

$$R_Z(t+1) \sim \mathrm{LN}[\nu_Z\Delta t, \sigma_Z^2(t)\Delta t] \sim \mathrm{LN}\{\nu_Z\Delta t, \mathrm{E}[\sigma_Z^2(t)]\Delta t\}, \tag{3.16}$$

the number of degrees of freedom, κ, increases by one:

$$\kappa(t+1) = \kappa(t) + 1; \tag{3.17}$$

and our new scaling factor becomes

$$\varsigma^2(t+1)\Delta t = \frac{\kappa(t)\varsigma^2(t)(\Delta t)^2 + [\ln(z_{t+1}) - \nu_Z \Delta t]^2}{\kappa(t)\Delta t + \Delta t} \tag{3.18}$$

(Gelman et al. 1995, pp. 46-8).

Example 3.4 (Learning About Volatility). Suppose we want to value an option C that pays Z dollars if exercised, where Z is an observable. The value of Z is seen and the options can be exercised at time 1 or time 2. Also suppose the current value of Z is 1, the return of Z has distribution

$$R_Z(t) \sim \mathrm{LN}[0.2, \sigma_Z^2(t)],$$

and we believe the squared volatility has distribution

$$\sigma_Z^2(t) \sim \text{Inv-}\chi^2[\kappa(t), \varsigma^2(t)],$$

where $\kappa(0) = 5$ and $\varsigma^2(0) = 0.2$. Assume the risk-free rate is 0.05.

From (3.16), we can write the distribution of the observable as

$$Z(t+1) = Z(t)\mathrm{LN}\left[\nu_Z(t), \frac{\kappa(t)\varsigma^2(t)}{\kappa(t)-2}\right].$$

We calculate the distribution of the observable at time 1 as

$$Z(1) \sim \mathrm{LN}\left(.2, \frac{1}{3}\right).$$

Using (3.17), we update the number of degrees of freedom of our estimate of the squared volatility: $\kappa(1) = 6$; and using (3.18), we update the scaling factor of our estimate of the squared volatility:

$$\varsigma^2(1) = \frac{1 + \{\ln[Z(1)] - .2\}^2}{6}.$$

Then we can calculate the distribution of the observable at time 2:

$$Z(2) \sim Z(1)\mathrm{LN}\left(\nu_Z(1), \frac{1 + \{\ln[Z(1)] - .2\}^2}{4}\right).$$

We exercise at the end of year one if, and only if, the expected return of period two of the observable is less than the risk-free return. That is, we exercise if

$$\begin{aligned}\hat{\mathrm{E}}[R_Z(2)] &= \exp\left\{\nu_Z(1) + \frac{1}{2}\mathrm{E}[\sigma_Z^2(1)]\right\} \\ &= \exp\left(.325 - \frac{9}{80}\sqrt{1 + \{\ln[Z(1)] - .2\}^2} + \{\ln[Z(1)] - .2\}^2\right) < e^{.05},\end{aligned}$$

which is never true. The value of the call option is thus:

$$C_0 = \int_0^\infty \int_0^\infty Z(2)\mathrm{d}Z(1)\mathrm{d}Z(2).$$

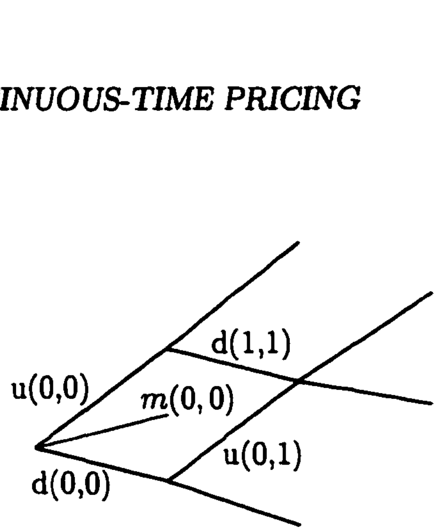


Figure 3.1: A learning lattice.

3.4.3 Learning Lattices

We next examine how to build a binomial lattice when we have an unknown expected growth rate. Because our estimate of the expected growth rate changes, we must force the observable's moves into a lattice as we build forward. Note that lattices for other unknown parameters can be built using techniques from this section, although these lattices may require a trinomial or higher order lattice.

Assume that we want to price an option on an observable Z. We believe the expected growth rate at a particular node has distribution

$$\nu_Z(a,t) \sim \mathrm{N}[m(a,t), \tau^2(t)],$$

where a is the number of up moves that have been made at the node and t is the time at the node. The return of Z at node(a,t) has distribution

$$R_Z(a,t) \sim \mathrm{LN}[\nu_Z(a,t), \sigma_Z^2] \sim \mathrm{LN}[m(a,t), \sigma_Z^2 + \tau^2(t)].$$

We build the learning lattice (see fig. 3.1) in three steps. We first build the middle section of the lattice, then the upper half of the lattice, and then the lower half of the lattice. The initial node has value $\ln(Z_0)$. After we find the first up and down moves, we set

$$\text{node}(1,2) = \ln(Z_0) + 2m(0,0)$$

to keep the lattice form. After we find the up and down moves from node(1,2), we set

$$\text{node}(2,4) = \ln(Z_0) + 4m(0,0).$$

We continue building in this manner until the entire middle section of the lattice is complete. Then, we build the upper half of the lattice by ensuring that the down moves reconnect with the lattice. In the lower half of the lattice, we ensure that the up moves reconnect with the lattice.

To find the size of our first up and down moves, we use the method of moments approximation (see sec. 4.2.1):

$$u(0,0) = m(0,0) + \sqrt{\sigma_Z^2 + \tau^2(0)}$$

and

$$d(0,0) = m(0,0) - \sqrt{\sigma_Z^2 + \tau^2(0)}.$$

To ensure that our branches reconnect at node(1,2), we set $d(1,1) = d(0,0)$ and $u(0,1) = u(0,0)$ (see fig. 3.1). Then we calculate the mean of our estimate of the expected growth rate using (3.15):

$$m(1,2) = m(0,0).$$

In general, for nodes$(t/2,t)$ and all even t (the entire middle section of the lattice),

$$m\left(\frac{t}{2},t\right) = m(0,0),$$

$$u\left(\frac{t}{2},t\right) = u\left(\frac{t}{2},t+1\right) = m(0,0) + \sqrt{\sigma_Z^2 + \tau^2(t)},$$

and

$$d\left(\frac{t}{2},t\right) = d\left(\frac{t}{2}+1,t+1\right) = m(0,0) - \sqrt{\sigma_Z^2 + \tau^2(t)}.$$

Notice in figure 3.1 that $u(1,1)$ and $d(1,1)$ make smaller absolute moves than $u(0,0)$ and $d(0,0)$ since $\tau^2(1) < \tau^2(0)$.

In the upper half of the lattice, we again need to set the down moves so that we keep a lattice form. This equates to setting:

$$d(a,t) = d(a-1,t-1) + u(a-1,t) - u(a-1,t-1).$$

We also update the mean of our expected growth rate:

$$m(a,t) = \frac{m(a-1,t-1)\sigma_Z^2 + \tau^2(t-1)u(a-1,t-1)}{\sigma_Z^2 + \tau^2(t-1)}.$$

With $d(a,t)$ and $m(a,t)$ in place, we solve the following set of equations:

$$\begin{aligned} p_u + p_d &= 1 \\ p_u u(a,t) + p_d d(a,t) &= m(a,t) \\ p_u u^2(a,t) + p_d d^2(a,t) &= \sigma_Z^2 + \tau^2(t) + m^2(a,t); \end{aligned}$$

to find

$$u(a,t) = m(a,t) + \frac{\sigma_Z^2 + \tau^2(t)}{m(a,t) - d(a,t)}.$$

(Smith (1990, pp. 43-7) shows that we can always solve this set of equations. More generally, he shows that we can almost always set one of j moves in advance and still match the first $2j-2$ moments.) We use a similar method to find the lower half of the lattice, which completes the development of the lattice.

When using this procedure, we obtain lattices that are similar to standard lattices. The main difference is that the probability of an up move sometimes becomes large in the top half of the lattice and small in the bottom half. If the variance in our estimate begins much larger than the squared volatility of the observable, these probabilities quickly grow close to one and zero. One way around this problem is to use larger time steps, but this leads to less accurate results. Another way is to not force the moves into a lattice for the first few time steps. This increases the number of nodes in the lattice, but the variance in our estimate quickly decreases to acceptable levels because the smaller the volatility is, the faster the variance in our estimate decreases. Another alternative is to use a trinomial lattice, which has more flexibility in setting the lattice parameters.

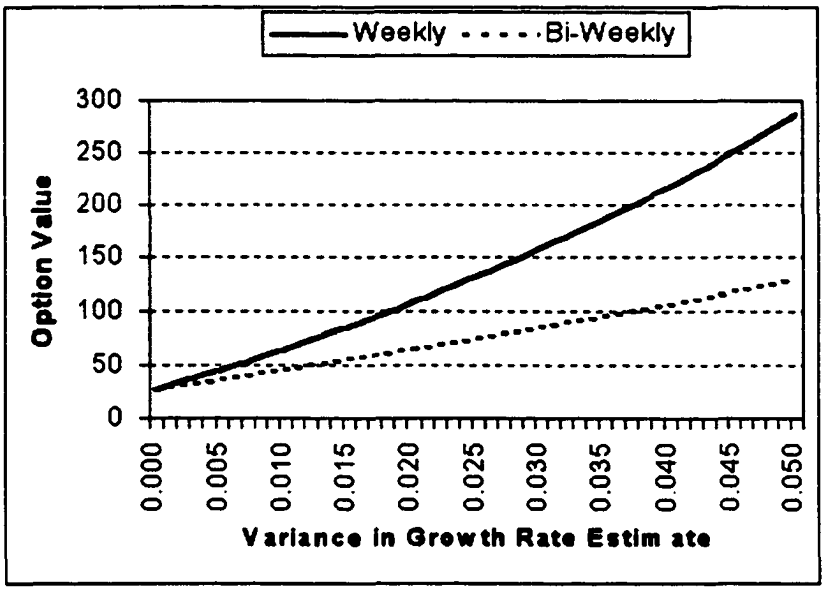


Figure 3.2: Weekly and bi-weekly learning.

Example 3.5 (Lattice Learning About Growth Rates). We want to price a one-year option on an observable Z. The current value of Z is 100 and the option pays $Z - 100$ dollars if exercised. We observe the value of Z at weekly intervals. The option can be exercised at any time. We believe that the volatility of Z is 0.5 and that the expected growth rate of Z has a Normal distribution with a mean of 0 and variance of τ^2. The risk-free rate is 0.05.

We approximate the value of the option with a binomial learning lattice with 52 time steps (i.e., $\Delta t = 1/52$). This option is priced for variances of our expected growth rate of $\tau^2(0) \in [0, .05]$ (see fig. 3.2). Notice the drastic change in the value of the option as the variance in our estimation of the growth varies. The options with lesser known growth rates are much more valuable since being uncertain about the expected growth rate is similar to increasing the observable's volatility. For comparison, we also price this option when we observe the value of Z at bi-weekly intervals (see fig. 3.2).

Through experimentation with this example, it appears as though continuous learning with a Normally-distributed growth rate leads to an infinitely-valued option.

3.5 Application — Development in Phases

A small firm has discovered a new technology that will allow the development of the next wave of hard disk drives. Around every six months, there is a new generation of hard disk drives. Unfortunately, the firm is understaffed and will be unable to develop the hard drive within six months; so, it must sell its technology. There is also a large company who, among other things, manufactures hard drives. Recently however, the large company has fallen behind with its technology and realizes that by the time the technology catches up, the company will have missed out on the latest generation. We seek to find the net present value of a possible synergy between the two firms. When trying to value this synergy, we find that the largest risk is development time. Traditionally and intuitively, this risk has been accounted for by simply taking the project manager's best guess of how many engineers are needed to maximize the chance of completing the project or getting to market on time. Unfortunately, this method undervalues the acquisition because it does not account for the option to abandon the project or to change the number of workers during the project.

We value this acquisition for the large manufacturer, and assume that the acquisition is small enough for us to be risk-neutral. Since other manufacturers will have the new hard drive ready in six months, and since this drive will only be sold for six months after that, it is important that we complete the development in time. We will make decisions on the number of engineers to use every quarter. Furthermore,

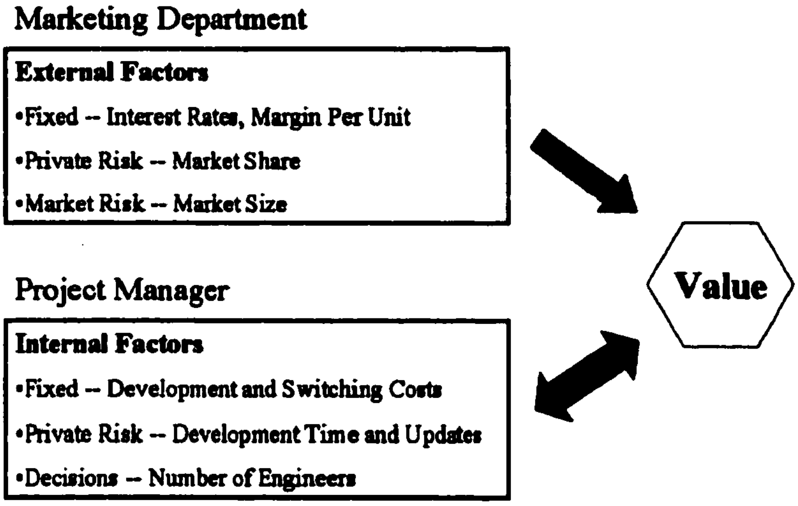


Figure 3.3: Forecasting needed.

we are able to divide the development into two distinct phases, the design phase and the prototype phase. Our goal is to solve for the optimal scheduling strategy and maximum buying price of the acquisition. To this end, we obtain information from the marketing department and the project manager.

Figure 3.3 shows what forecasts need to be made and by whom. It also describes which factors are fixed, *private risk* (risk that cannot be hedged by buying or selling any market asset), and *market risk* (risk that can be hedged by buying or selling a market asset). The arrows show that the value of the acquisition depends on both the external and internal factors and that the decision on the number of engineers depends on the value of the acquisition. Thus, the decision on the number of engineers is indirectly influenced by the external factors.

3.5.1 Marketing Department

Our marketing department believes that the market size for hard drives has an expected growth rate of 25% and a volatility of 30%. The market size has a 70% correlation with a basket of personal computing stocks, which has an expected growth rate of 15% and an implied volatility – as seen from options on the market – of 30%. There were 800,000 hard drives sold last quarter. The marketing department estimates that if the new hard drive is ready within six months, our market share will be about 50% for the remaining two quarters. If the product is delivered one quarter late, our market share will be about 20% for the remaining quarter. No sales will be made after the fourth quarter. The department also believes that our market share is uncorrelated with the Market portfolio and that it should be modeled as a random variable that has an equal chance of rising or falling by five points each quarter. Historically, the margin on each hard drive sold has been ten dollars and the department expects no change. The risk-free rate is 10% and we assume that it will remain constant throughout the life of the project.

3.5.2 Project Manager

Our project manager estimates how many quarters it will take various numbers of engineers to complete the design phase in the best, average, and worst circumstances in table 3.1. For example, table 3.1 shows that twenty engineers will take two quarters to complete the design phase in the worst case. We assume that this implies that one-half of the design phase will be completed after one quarter. If we again use twenty engineers in the second quarter, then in the best case we will complete the design phase and one-half of the prototype phase. In general, after the design phase

Table 3.1: Quarters to Complete Design Phase

Number of Engineers	Best	Average	Worst
Five	2	3	3
Ten	1	2	3
Fifteen	1	2	2
Twenty	1	1	2
Twenty-Five	1	1	2

Table 3.2: Quarters to Complete Prototype Phase

	Expected			Short			Long		
Engineers	Best	Ave	Worst	Best	Ave	Worst	Best	Ave	Worst
Five	1	2	2	1	1	1	2	3	3
Ten	1	1	2	1	1	1	1	2	3
Fifteen	1	1	1	1	1	1	1	1	2
Twenty	1	1	2	1	1	1	1	2	3
Twenty-Five	1	1	2	1	1	1	1	2	3

is complete, a more accurate estimate of the number of quarters to complete the prototype phase will be given. Table 3.2 shows the updated prototype phase. There is a 50% chance that there will be no change in the estimate, a 25% chance that it will be lengthened, and a 25% chance that it will be shortened. The manager believes that the probability of a best or worst outcome in any quarter and any phase is 25% each, and the probability of an average outcome is 50%. We can see from table 3.1 and table 3.2 that there are decreasing returns with the number of engineers. Thus, the intuitive strategy is to use twenty engineers in the design phase and fifteen engineers in the prototype phase. This strategy maximizes the chance of completing the project on time (62.81%) and minimizes the chance of not completing it at all (0.55%). We

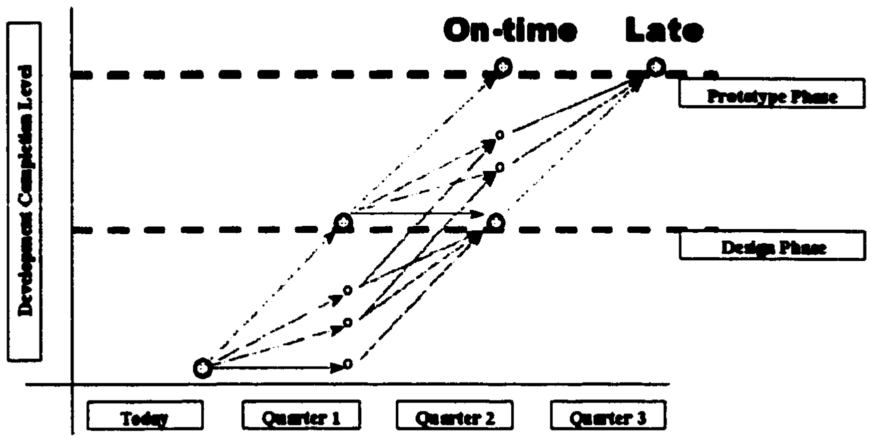


Figure 3.4: The development process.

compare this strategy to the optimal strategy in the next section. The number of engineers can be changed at the end of each quarter, but the quarter immediately following a change has a 15% chance of the best outcome and a 35% chance of the worst outcome. The cost per engineer per quarter is $80,000 paid at the beginning of the quarter.

3.5.3 Strategy and Valuation

An outline of the development process is shown in figure 3.4. We model market size as a binomial lattice. The risk-neutral growth rate of the market size Z is found from equation G–2(3.10) as

$$\begin{aligned}\nu_{Z_Q} &= \nu_Z - \beta_{Z,S}\left(\nu_S + \frac{1}{2}\sigma_S^2 - r_f\right)\\ &= .25 - \frac{.7(.3)}{.3}\left[.15 + \frac{1}{2}(.09) - .1\right] = .1835,\end{aligned}$$

where S is the price of the basket of personal computing stocks correlated with the market size. Using the binomial log-transform lattice(BI) (see sec. 4.2.1) with a time step of one quarter, the chance of moving up or down is one-half, the up move of the lattice is

$$\ln(u) = \nu_{Z_Q}\Delta t + \sigma_Z\sqrt{\Delta t} = \frac{.1835}{4} + \frac{.3}{2} = .1959,$$

and the down move is

$$\ln(d) = \nu_{Z_Q}\Delta t - \sigma_Z\sqrt{\Delta t} = \frac{.1835}{4} - \frac{.3}{2} = -.1041.$$

We model market share as a binomial lattice that moves up or down five points per quarter with equal chance. To solve with dynamic programming, we solve for the optimal strategy at the end of quarter two; knowing what our strategy will be then, we solve for the optimal strategy at the end of quarter one; knowing that, we solve for the current optimal strategy. To find the optimal strategy at each time point, we consider which phase we are in (design or prototype), the amount of work left to complete the phase, the length of the phase (if prototype), the market share, the market size, and the number of engineers we used last quarter.

The optimal strategy is to begin with twenty engineers. If the design phase is not completed during the first quarter, however, we will stop development unless both the market size and market share have moved favorably, in which case we will use fifteen engineers in the second quarter. If the design phase is completed in the first quarter and the project manager feels that the prototype phase will be short, then we will use five engineers in the second quarter; if he feels that the prototype phase

Table 3.3: End of Quarter Two Engineers

Work Left	Update	Sh D Si U	Sh M Si U	Sh U Si U	Sh D Si M	Sh M Si M	Sh U Si M	Sh D Si D	Sh M Si D	Sh U Si D
$> .5$	Short	5	5	5	5	5	5	5	5	5
$> .5$	None	0	15	15	0	15	15	0	15	15
$> .5$	Long	0	15	15	0	0	15	0	0	15
$\leq .5$	Short	5	5	5	5	5	5	5	5	5
$\leq .5$	None	5	5	5	5	5	5	5	5	5
$\leq .5$	Long	0	15	15	0	15	15	0	15	15

Notes: 'Sh' represents Market Share, 'Si' represents Market Size, 'U' represents Up, 'M' represents Middle, 'D' represents Down, and 'Work Left' represents the amount of development remaining to complete the prototype phase.

will be long or as previously expected, we will use fifteen engineers. If the prototype phase is not completed by the end of quarter two, then the future strategy will vary as a function of the amount of work left, the prototype update, the market size, and the market share (see table 3.3).

A comparison of the intuitive and optimal strategies is shown in figure 3.5. Both the optimal and intuitive strategies result in a 62.81% chance of completing the project on time. The difference between the two strategies is in the chance of abandoning the project as opposed to completing the project late. The optimal strategy has a 9.03% chance of abandonment, and the intuitive strategy has only a .55% chance of abandonment. The higher rate of abandonment lowers costs and makes the acquisition 19.04% more valuable with the optimal strategy than with the intuitive strategy. The maximum buying price of the acquisition is $3.37 million.

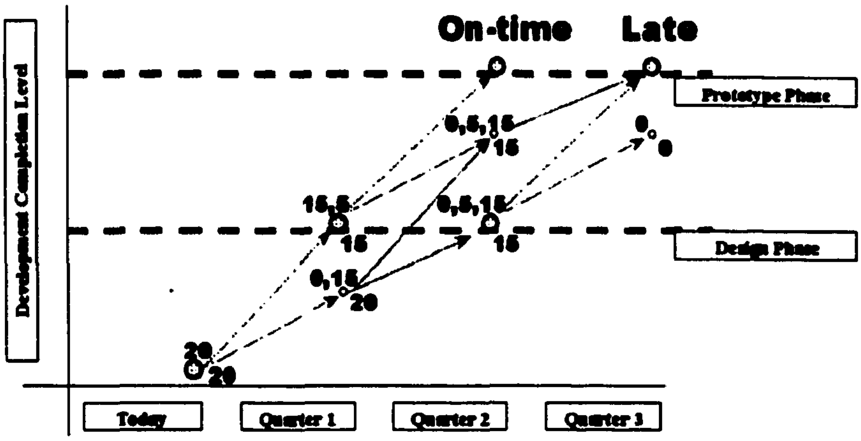


Figure 3.5: The optimal number of engineers varies and is shown above the nodes. The intuitive number of engineers is shown below the nodes.

Chapter 4

Lattices

To solve for the value of more complex derivatives, numerical methods are used. The most important attribute of these methods is whether a risk-neutral valuation technique can be used to simplify the calculations and to transform variables so that only one discount rate is necessary (see, e.g., Luenberger (1998a, pp. 251-3) and Duffie (1996, p. 91)).

The values of complex derivatives in which the strategy is known in advance are approximated by *Monte Carlo Simulation*, a simple and flexible method for pricing European-style derivatives. Monte Carlo Simulation uses random numbers to sample many different paths that the underlying variables could follow under risk-neutral probabilities or growth rates. Each path's payoff is discounted at the risk-free rate and the average of these payoffs is the estimated value of the derivative. It is powerful when there are many underlying variables or path-dependent payoffs.

When the strategy is not known in advance, we break up the moves of the underlying asset into many pieces and form a grid of nodes that represents various asset prices at a finite number of time points. The value of the derivative is found with a *dynamic-programming*-type procedure, which is a backward-looking procedure as opposed to a forward-looking procedure like Monte Carlo Simulation. Backward-looking means that the value of the derivative is found by first calculating the value at the end of the derivative's life and then working backward. There are two major types of methods that use grids, *finite difference* methods and *lattice* methods.

A finite difference method values derivatives by numerically solving the differential equation that the derivative satisfies. The equation is converted into a set of difference equations and is solved with dynamic programming. The largest benefit of a finite difference method is that it is the only method that handles many different starting values at once. It is arguably the most complex and least intuitive of the methods, however. In addition, for some derivatives, especially those that have multiple interacting options, we cannot form an explicit set of partial differential equations (see, e.g., Trigeorgis (1996, pp. 305-6) and Hull (1993, pp. 352-62)).

A *tree* consists of a set of nodes with time represented along the horizontal and a variable's value represented along the vertical. A binomial tree starts out as a single node, which represents a variable's value today, and branches into two new nodes that represent the variable's value at the next time point; then each of these nodes branches into two new nodes and so on. A lattice is a tree that recombines. For example, an up move followed by a down move leads to the same variable value as a down move followed by an up move. Lattices contain significantly fewer nodes than other trees for a given number of time steps.

The most commonly used lattice for representing stock prices is the multiplicative binomial lattice of Cox, Ross, and Rubinstein (CRR) (Cox, Ross, and Rubinstein 1979). The CRR Method begins with a discrete-time approximation of geometric

Brownian motion and then builds a lattice. It then finds the risk-neutral probabilities of the lattice for proper pricing. The CRR Lattice's weaknesses are it is only capable of handling one underlying asset at a time and it does not price accurately for all parameters. For a more detailed comparison of the CRR Method, finite difference method, and Monte Carlo simulation; see Geske and Shastri (1985).

There are many ways to build lattices for representing a variable's values. Several lattices have been built that approximate a risk-neutral lognormal diffusion model (Log-Transform Model) (Trigeorgis 1991; Omberg 1988; Jarrow and Rudd 1983, pp. 183-6). The method of Omberg (1988) approximates the Log-Transform Model by matching moments of the Normally-distributed Brownian motion. A moment-matching method approximates a continuous distribution with a discrete one by matching as many consecutive moments as possible starting from the zeroth moment. These lattices can be of any order (e.g., Binomial(BI) or Trinomial(TRI)). To make the lattices more efficient and to increase the accuracy around the optimal exercise point, Omberg (1988) also builds lattices by using equal spacing between nodes (with spacing equal to the spacing of the innermost node found in the pure moment matching method). Jarrow and Rudd (1983, pp. 183-6) also build the BI. Tian (1993) tests a lattice he builds from matching the moments of the lognormal distribution of the underlying asset (see Easton (1996) for a note on the test results).

Real options models typically contain a web of interacting compound options, have several underlying variables, and are quite complex. Lattices are arguably the most useful tools for valuing many real options because they are simple, intuitive, flexible, and able to handle early exercise and many types of underlying stochastic processes.

4.1 How to Compare

In this dissertation, we seek a simple lattice method that gives an accurate approximation of the value of myriad options that contain at least one discontinuity. More specifically, we want the following three attributes:

Simplicity A simple method is intuitive and easy to understand and use.

Accuracy An accurate method prices options close to the real value with few calculations. We will define four types of accuracy: *Distribution Accuracy* is the accuracy of the approximating distribution of the underlying variable at any given time point. With more nodes at a given time point, we more accurately pinpoint the optimal exercise price. *Time Accuracy* is the accuracy with which we model the time line. With more time steps, we more accurately pinpoint the time to exercise an option. *Standard Accuracy* is the pricing accuracy of the lattices in standard error statistics. *Percentage Accuracy* is the pricing accuracy of the lattices in percentage error statistics.

Robustness A robust method is consistently stable, accurate, and simple for both univariate and multivariate options with a wide range of parameters. A *stable* method's discrete-time approximation converges exactly to its continuous model as we shrink the time step to zero. According to Lindeberg's Central Limit Theorem, for a lattice to have stability when approximating Brownian motion, it is sufficient that each probability is between zero and one; the zeroth, first, and second moments are matched; and the moves are independent of the underlying variable's values (Tian 1993). To have stability when approximating multiple Brownian motions, we must also match the correlations between variables. If the method is stable for all parameters, it is *unconditionally stable.* If a method is stable, then the expectation of a terminal payoff in discrete time will converge to the continuous time one. For conditions and proofs of convergence of more general strategies (e.g., path-dependent strategies), see Kushner and Dupuis (1992).

4.2 Log-Transform Model

Let $\Delta t = T/n$ be the length of each time step, where T is the total time and n is the number of time steps. To approximate the value of continuous-time options on an observable Z with a lattice, the dynamics of (3.4) are converted into the discrete-time model,

$$\Delta \ln(Z) = \nu_{Z_Q}\Delta t + \sigma_Z\sqrt{\Delta t}\Delta \hat{\mathrm{B}},$$

where ν_{Z_Q} is the risk-neutral expected growth rate of Z, σ_Z is the risk-neutral volatility of Z (same as the original volatililty of Z), and $\hat{\mathrm{B}}$ is a risk-neutral Brownian motion. We first approximate $\Delta\hat{\mathrm{B}}$ using a pure moment-matching method. Then we use a moment-matching method that requires equal spacing between nodes. And finally we use a moment-matching method that requires both equal spacing and equal probabilities. Each of these methods is extendable to multiple lattices, unconditionally stable for both single and multiple lattices, and easy to understand. Of all the lattices tested, the binomial lattice formed out of the Log-Transform model by pure moment matching (Log-Transform Binomial Lattice (BI)) has the greatest Standard Accuracy and the Log-Transform Trinomial Lattice (TRI) has the greatest Percentage

Accuracy (see sec. 4.4 for detailed test results), and both are easy to use.

4.2.1 Pure Moment Matching

The pure moment-matching method matches as many moments of $\Delta\hat{\mathrm{B}}$ as possible. Since the Log-Transform Quadrinomial (Quadrinomial) and higher-order lattices have more than one spacing size, they are more difficult to use, have an increased number of nodes per time step, and have decreased pricing accuracy. As the number of nodes in the lattices is increased, the Distribution Accuracy increases faster than the Time Accuracy, which could be the reason for their decreased pricing accuracy.

The kth moment is defined as $\mathrm{E}(x^k)$ (see Billingsley (1995) and Johnson and Kotz (1972) for more information on moments). There are restrictions on what the moments of a probability distribution can be; for example, the variance-covariance matrix should be positive semi-definite (see Smith (1990, p. 22) for other examples). We must be careful because these restrictions are sometimes violated when using estimations of moments (especially correlations).

To use moment matching to approximate $\Delta\hat{\mathrm{B}}$ by a tree with j possible moves at each node, we solve the following $2j$ equations for the unknown variables X_i and p_i:

$$\sum_{i=1}^{j} p_i X_i^k = \mathrm{E}(x^k), \quad \text{for } k = 0, 1, \ldots, 2j-1, \tag{4.1}$$

where X_i is the size of the move and p_i is the probability of the ith move occuring.

Using the method of Smith (1990, p. 37), we can always find a tree with j possible moves at each node by matching the first $2j-1$ moments, that is, we can always solve (4.1) with nonnegative probabilities. This tree underestimates the magnitude of all even moments of order greater than or equal to $2j$ (Smith 1990, p. 59).

We know $\Delta\hat{\mathrm{B}}$ is Normally distributed with mean 0 and variance $\sigma_Z^2\Delta t$. The BI assumes that in each short interval of time, Δt, $\ln(Z)$ either moves up by u or moves down by d. Let q_u represent the risk-neutral probability of making an up move of size u, and let q_d represent the risk-neutral probability of making a down move of size $d < u$. To find the parameters of the BI, solve (4.1) for $k = 0, 1, 2, 3$:

$$\begin{aligned}
q_u + q_d &= 1 \\
q_u u + q_d d &= \nu_{Z_Q} \\
q_u u^2 + q_d d^2 &= \nu_{Z_Q}^2 + \sigma_Z^2 \\
q_u u^3 + q_d d^3 &= \nu_{Z_Q}^3 + 3\nu_{Z_Q}\sigma_Z^2.
\end{aligned}$$

The risk-neutral probability of an up or down move is $1/2$, the up move is $u = \nu_{Z_Q}\Delta t + \sigma_Z\sqrt{\Delta t}$, and the down move is $d = \nu_{Z_Q}\Delta t - \sigma_Z\sqrt{\Delta t}$. The fourth moment of the approximation is $2\sigma_Z^4$ less than the fourth moment of the Normal distribution, which is $3\sigma_Z^4$.

As we will see in chapter 5, the BI is extendable to multiple lattices. Jarrow and Rudd (1983, pp. 183-6) and Omberg (1988) also constructed the BI. In addition, Omberg (1988) uses pure moment matching to build the TRI and other lattices out of the Log-Transform Model.

Probabilities and moves are shown in table 4.1 for the TRI, Quadrinomial, and Log-Transform Quintinomial (Quintinomial) lattices. Table 4.1 shows the largest absolute move of the lattice on the left and then moves progressively towards the smallest absolute move. For example, the TRI can make three moves: up u, down d,

Table 4.1: Moment-Matching Lattices

	TRI	Quadrinomial	Quintinomial
Probability	$\frac{1}{6}, \frac{2}{3}$	$(3-\sqrt{6})/12, (3+\sqrt{6})/12$	$\frac{7-2\sqrt{10}}{60}, \frac{7+2\sqrt{10}}{60}, 8/15$
Spacing	$\sqrt{3}, 0$	$\sqrt{1+\sqrt{5+2\sqrt{6}}}, \sqrt{1-\sqrt{5-2\sqrt{6}}}$	$\sqrt{5+\sqrt{10}}, \sqrt{5-\sqrt{10}}, 0$

Note: The moves are centered around the expected growth rate, $\nu_{Z_Q}\Delta t$, and are in terms of $\sigma_Z\sqrt{\Delta t}$.

or middle m. The probability of either an up or down move is $1/6$ and the probability of a middle move is $2/3$. The TRI matches the first five moments, but undervalues the sixth moment of the Normal distribution, which is $15\sigma_Z^6$, by $6\sigma_Z^6$. The Quadrinomial can make four moves: a large up move, a small up move, a small down move, or a large down move. The Quadrinomial performed poorly under testing. The Quintinomial, which has five possible moves, also has two spacing sizes but was not tested since it is difficult to use and since the Quadrinomial had poor results.

4.2.2 Equal Spacing

One way to ensure that a lattice recombines is to force the moves to be equally-spaced. We build an Equal-Spacing lattice by finding the lattice that has equal spacing between moves, matches as many moments as possible, and is as close as possible to matching the next moment. With the equal-spacing constraint, however, we match fewer moments. For example, the Equal-Spacing Quadrinomial Lattice matches three moments instead of the Quadrinomial's seven, and the Equal-Spacing Quintinomial Lattice matches five moments instead of the Quintinomial's nine.

Table 4.2: Equal-Spacing Lattices

	Quadrinomial	Quintinomial
Probability	$.05, .45$	$1/75, 16/75, 41/75$
Spacing	$\sigma_Z\sqrt{20\Delta t/9}$	$\sigma_Z\sqrt{15\Delta t/8}$

Note: The moves are centered around the expected growth rate, $\nu_{Z_Q}\Delta t$.

The probabilities and spacing for the Equal-Spacing Quadrinomial and Equal-Spacing Quintinomial lattices are shown in table 4.2. Equal-Spacing Lattices are easy to use and have a lot of Percentage Accuracy; however, the pricing accuracy drops with higher order lattices. Because of the equal spacing, the Time Accuracy and Distribution Accuracy increase at the same rate, which leads to greater pricing accuracy. In particular, the Equal-Spacing Quadrinomial Lattice has a good balance of both Standard Accuracy and Percentage Accuracy, and the Equal-Spacing Quintinomial has less pricing accuracy than the Equal-Spacing Quadrinomial or than the TRI.

4.2.3 Equal Probability and Equal Spacing

Another way to design a lattice is to equate the spacing and to equate the probabilities. For example, the Equal-Probability Trinomial has a one-third chance of moving up, down, or middle. The parameters of these lattices are easy to find since we are limited to matching the first three moments. The first and third moments are matched by centering the spacing around the mean and the second moment is matched by choosing the step size. Equal-Probability lattices, which include the BI, have a lot of Standard Accuracy; however, pricing accuracy drops with higher order lattices. The

Table 4.3: Equal-Probability Lattices

	Trinomial	Quadrinomial	Quintinomial
Spacing	$\sigma_Z\sqrt{3\Delta t/2}$	$\sigma_Z\sqrt{4\Delta t/5}$	$\sigma_Z\sqrt{\Delta t/2}$

Note: The moves are centered around the expected growth rate, $\nu_{Z_Q}\Delta t$.

Table 4.4: Comparisons of Log-Transform Methods

	Moments Matched	Standard Accuracy	Percentage Accuracy	Ease of Use
BI	3	High	Medium	High
TRI	5	Low	High	High
Equal-Probability Trinomial	3	Medium	Medium	High
Equal-Spacing Quadrinomial	3	Medium	Medium	Medium
Equal-Probability Quadrinomial	3	Medium	Low	Medium
Equal-Spacing Quintinomial	5	Medium	Low	Medium
Equal-Probability Quintinomial	5	Medium	Low	Medium
Quadrinomial	7	Low	Low	Low
Quintinomial	9	Low	Low	Low

spacings for the Equal-Probability Trinomial, Equal-Probability Quadrinomial, and Equal-Probability Quintinomial lattices are shown in table 4.3.

4.2.4 Comparisons

Table 4.4 compares the Log-Transform Lattices in number of moments matched, Standard Accuracy, Percentage Accuracy, and ease of use.

To help understand why some lattices perform better than others, the asymptotic limits of the number of time steps, end nodes (nodes at the last time step), average spacing between nodes, and total range of end nodes were calculated and are shown in table 4.5. It is reasonable to assume that having a large range results in more

Table 4.5: Asymptotic Comparisons

	Time Steps	End Nodes	Average Spacing	Total Range
BI	$\sqrt{2z}$	$\sqrt{2z}$	$\sqrt[4]{8/z}$	$2\sqrt[4]{2z}$
TRI	$\sqrt{z}$	$2\sqrt{z}$	$\sqrt[4]{9/z}$	$2\sqrt[4]{9z}$
Equal-Probability Trinomial	$\sqrt{z}$	$2\sqrt{z}$	$\sqrt[4]{9/(4z)}$	$2\sqrt[4]{9z/4}$
Equal-Spacing Quadrinomial	$\sqrt{2z/3}$	$\sqrt{6z}$	$\sqrt[4]{200/(27z)}$	$2\sqrt[4]{50z/3}$
Equal-Probability Quadrinomial	$\sqrt{2z/3}$	$\sqrt{6z}$	$\sqrt[4]{24/(25z)}$	$2\sqrt[4]{54z/25}$
Equal-Spacing Quintinomial	$\sqrt{z/2}$	$2\sqrt{2z}$	$\sqrt[4]{225/(32z)}$	$2\sqrt[4]{225z/8}$
Equal-Probability Quintinomial	$\sqrt{z/2}$	$2\sqrt{2z}$	$\sqrt[4]{1/(2z)}$	$2\sqrt[4]{2z}$
Quadrinomial	$\sqrt[3]{3z}$	$2.08\sqrt[3]{z^2}$	$2.35\sqrt{1/z}$ *	$4.89\sqrt[6]{z}$ *
Quintinomial	$\sqrt[3]{3z/2}$	$2.62\sqrt[3]{z^2}$	$2.33\sqrt{1/z}$ *	$6.11\sqrt[6]{z}$ *

Note: z is the number of nodes in the lattice.

* Unequal spacing between nodes.

Percentage Accuracy since the lattice is more accurate at the edges of the underlying variable's distribution. It is also reasonable to assume that having tight spacing between nodes results in more Standard Accuracy since for most options the lattice is more precise around the optimal exercise point. And, it is reasonable to assume that having a large number of time steps plays a role in both of these types of pricing accuracies since the Time Accuracy is greater.

4.3 Other Models

In this section, we look at the binomial lattice version of several other types of models that use moment-matching methods and at the most widely used lattice model, the Cox, Ross, and Rubinstein Binomial Lattice.

4.3.1 Cox, Ross, and Rubinstein Model

The CRR Model (Cox, Ross, and Rubinstein 1979) is inaccurate and sometimes unstable. The CRR Lattice is built through a two-step process, and its pricing accuracy depends on the expected growth rate of an asset S, ν_S, although the actual price of the option does not depend on the expected growth rate.

The first step begins with the discrete-time model,

$$\Delta \ln(S) = \nu_S \Delta t + \sigma_S \sqrt{\Delta t} \Delta \mathrm{B}.$$

The first-step parameters are found by matching the zeroth, first, and second moment of $\Delta \mathrm{B}$, and setting $u = -d$:

$$p_u = \frac{1}{2} + \frac{\nu_S}{2\sqrt{\frac{\sigma_S^2}{\Delta t} + \nu_S^2}}$$

and $u = \sqrt{\sigma_S^2 \Delta t + \nu_S^2 (\Delta t)^2}$.

This solution is not risk-neutral and therefore not appropriate for pricing. In step two, we find the risk-neutral probabilities for our pricing lattice. From Q–1(2.14), $R_f = q_u \mathrm{e}^u + q_d \mathrm{e}^d$, where $R_f = \mathrm{e}^{r_f}$ is the risk-free return. By replacing q_d with $1 - q_u$, we see that the risk-neutral probability of an up move is

$$q_u = \frac{R_f - \mathrm{e}^d}{\mathrm{e}^u - \mathrm{e}^d}. \tag{4.2}$$

For stability, we see from (4.2) that we must have $u \geq r_f \Delta t \geq d$, which is expanded and rearranged to show that we must have

$$n \geq \frac{T(r_f^2 - \nu_S^2)}{\sigma_S^2}.$$

Notice that the risk-neutral lattice only matches the zeroth and first moments until the limit.

This method is complicated because of the two-step process and the two sets of probabilities. Several authors present methods that extend this model to multiple lattices (Luenberger 1998b; Kamrad and Ritchken 1991; He 1990; Madan, Milne, and Shefrin 1989; Boyle, Evnine, and Gibbs 1989; Boyle 1988).

4.3.2 Alternative Log-Transform

The Log-Transform Model does not perfectly match the risk-neutral expected growth rate except in the limit (Nawalkha and Chambers 1995). The Alternative Log-Transform Model (Alternative) is similar to the Log-Transform Model but matches the risk-neutral expected growth rate exactly, which makes it a better model when looking for optimal strategy rules. This model, however, is more complicated and has less pricing accuracy than the Log-Transform Model. In the Alternative Log-Transform Model, the risk-neutral probabilities, q_i, are the same as in the Log-Transform Model but the moves, X_i^*, are different than the Log-Transform moves, X_i:

$$X_i^* = X_i - \frac{1}{2}\sigma_Z^2 \Delta t + \ln\left[\sum_{j=1}^{n} q_j \exp(X_j - \nu_{Z_Q}\Delta t)\right].$$

For example, the Alternative Log-Transform Binomial Lattice has up move

$$u = \left(\nu_{Z_Q} - \frac{1}{2}\sigma_Z^2\right)\Delta t + \sigma_Z\sqrt{\Delta t} + \ln\left(\frac{e^{\sigma_Z\sqrt{\Delta t}} + e^{-\sigma_Z\sqrt{\Delta t}}}{2}\right).$$

4.3.3 Multiplicative

The Multiplicative Model is simple but is Normally distributed instead of lognormally distributed. Compared to the other models, it has little pricing accuracy when there are only a few time steps but it has a medium amount of pricing accuracy when the number of time steps is large. This model converts the risk-neutral version of (3.2) into the discrete-time model,

$$R_Z = 1 + \left(\nu_{Z_Q} + \frac{1}{2}\sigma_Z^2\right)\Delta t + \sigma_Z\sqrt{\Delta t}\Delta\hat{\mathrm{B}}.$$

To make this model exactly match the risk-neutral expected growth rate, we approximate $1 + (\nu_{Z_Q} + \sigma_Z^2/2)\Delta t$ with $\exp[(\nu_{Z_Q} + \sigma_Z^2/2)\Delta t]$, which results in the following process:

$$R_Z = \exp\left[\left(\nu_{Z_Q} + \frac{1}{2}\sigma_Z^2\right)\Delta t\right] + \sigma_Z\sqrt{\Delta t}\Delta\hat{\mathrm{B}}.$$

We approximate $\Delta\hat{\mathrm{B}}$ by matching moments. For example, the Multiplicative Binomial Lattice has *returns*,

$$U = \exp\left[\left(\nu_{Z_Q} + \frac{1}{2}\sigma_Z^2\right)\Delta t\right] + \sigma_Z\sqrt{\Delta t}$$

and

$$D = \exp\left[\left(\nu_{Z_Q} + \frac{1}{2}\sigma_Z^2\right)\Delta t\right] - \sigma_Z\sqrt{\Delta t},$$

and risk-neutral probabilities, $q_U = 1 - q_D = \frac{1}{2}$.

4.3.4 Lognormal

The Lognormal Model starts with the same discrete-time model as the Log-Transform Model but then takes the exponent of both sides to get

$$R_Z = \exp\left(\nu_{Z_Q}\Delta t + \sigma_Z\sqrt{\Delta t}\Delta\hat{\mathrm{B}}\right).$$

Then, instead of matching the moments of $\Delta\hat{\mathrm{B}}$, it matches the moments of

$$\exp\left(\sigma_Z\sqrt{\Delta t}\Delta\hat{\mathrm{B}}\right).$$

This model is powerful in theory, but is difficult to use for pricing since the limitation is the accuracy of computing exponents to small powers. It is also complicated to find the appropriate parameters. For single underlying derivatives, we use a moment-matching method proposed by Smith (1990). But for multivariate derivatives, it is difficult. We find the returns, X (U and D), and the risk-neutral probability, q, with the binomial moment-matching solution for the lognormal distribution:

$$X = \frac{\exp\left(\nu_{Z_Q} + \frac{3}{2}\sigma_Z^2\right)}{2(e^{\sigma_Z^2} - 1)}\left(e^{2\sigma_Z^2} - 1 \pm \sqrt{e^{4\sigma_Z^2} - 6e^{2\sigma_Z^2} + 8e^{\sigma_Z^2} - 3}\right)$$

Table 4.6: Summary of Models

	Standard Accuracy	Percentage Accuracy	Unconditionally Stable	Robust	Simple
Log-Transform	High	High	Yes	Yes	Yes
CRR	Low	Low	No	No	No
Alternative	Medium	Medium	Yes	Yes	No
Multiplicative	Medium	Medium	Yes	No	Yes
Lognormal	Low	Low	Yes	No	No

and

$$q = \frac{1}{2} \mp \frac{2 + e^{3\sigma_z^2} - 3e^{\sigma_z^2}}{2e^{\sigma_z^2}\sqrt{e^{4\sigma_z^2} - 6e^{2\sigma_z^2} + 8e^{\sigma_z^2} - 3}}.$$

Tian (1993) also builds and tests this model. See Easton (1996) for a note on the test results.

4.3.5 Model Comparisons

Table 4.6 compares the models we have covered in Standard Accuracy, Percentage Accuracy, stability, robustness, and simplicity. The Log-Transform Model has the most pricing accuracy and is the only model that is unconditionally stable, robust, and simple.

4.4 Testing

To gain insight into how these new lattices would perform with complex real options, a comprehensive test was designed that tests lattices on options over a wide range of parameters. There were 320 American and 320 European put options valued by BI, TRI, Quadrinomial, Equal-Spacing Quadrinomial, Equal-Spacing Quintinomial, Equal-Probability Trinomial, Equal-Probability Quadrinomial, Equal-Probability Quintinomial, CRR Lattice with an expected growth rate of 0.15, CRR Lattice with an expected growth rate of 0.5, Alternative Log-Transform Binomial Lattice, and Multiplicative Binomial Lattice. The Quintinomial Lattice was excluded since it is difficult to use and since the Quadrinomial performed poorly; the Lognormal Model was excluded because of errors in calculating exponents correctly. In addition, for a benchmark, the European options were priced using an analytical approximation of the Black-Scholes formula, which is accurate to about six decimal points (Luenberger 1998a, p. 378). The starting value of the underlying variable was always 100 and the other parameter values were from the following set:

$$\begin{aligned} \nu_{Z_Q} &\in \{0, 0.05, 0.1, 0.15\} \\ \sigma_Z &\in \{0.1, 0.4, 0.7, 1\} \\ T &\in \{0.01, 0.1, 1, 10\} \\ K &\in \{70, 80, 100, 130, 170\}; \end{aligned}$$

where T is the option's time length and K is the strike price. Every combination of these parameters' values were used as one of the 320 options ($4\times4\times4\times5 = 320$). After finding the value of the options, we calculated the average error (AVE), average absolute error (AAE), average squared error (ASE), maximum absolute error (MAXE), average percentage error (%AVE), average absolute percentage error (%AAE), average squared percentage error (%ASE), and maximum absolute percentage error (%MAXE). Because the statistics based on percentage errors are unstable for small

Table 4.7: Time Steps per Nodes Comparisons

	1,000,000 Nodes	10,400 Nodes	1000 Nodes	239 Nodes
BI	1413	143	43	20
TRI	999	101	31	14
Equal-Probability Trinomial	999	101	31	14
Equal-Spacing Quadrinomial	816	82	25	12
Equal-Probability Quadrinomial	816	82	25	12
Equal-Spacing Quintinomial	700	71	22	10
Equal-Probability Quintinomial	700	71	22	10
Quadrinomial	143	30	13	7
Quintinomial	113	24	10	6

values, the options that had a Black-Scholes value of less than 0.01 were thrown out when calculating percentage errors. There were 270 options remaining.

The computational time for American options was proportional to the total number of nodes in the lattice at about 3.7×10^{-5} seconds per node (VBA for Excel, 266MHz Pentium II, 64MB RAM). Because of computer memory constraints, lattices with more than one-million nodes slowed the calculation time considerably. The unequal spacing lattices took about 25% longer than other lattices and reached memory capacity sooner. The number of nodes used for testing were 1,000,000; 10,400; 1,000; and 239. These numbers of nodes are approximately the number of nodes needed for the Quadruple, Triple, Double, and BI to take twenty time steps (see ch. 5 for multiple lattices). For each of these numbers of nodes, the number of time steps each Log-Transform lattice makes is shown in table 4.7.

Exploratory tests showed that the BI has the greatest Standard Accuracy and that the TRI has the greatest Percentage Accuracy for American options. Therefore, we will use the BI with one-million nodes as our benchmark for errors and the TRI with one-million nodes as our benchmark for percentage errors for American options. All TRI American option values less than 0.01 were thrown away for percentage error statistics. There were 265 options left. Figure 4.1 compares all twelve tested lattice types at 239 nodes. Based on the test results, the BI should be the lattice that is most commonly used and the TRI should be the lattice that is used for options with extreme strike prices. Notice that the Equal-Probability Trinomial and Equal-Spacing Quadrinomial are ranked among the top four lattices with respect to both Standard Accuracy and Percentage Accuracy.

Figure 4.2 shows the test results of the BI and TRI with approximately one-million nodes versus the Black-Scholes approximation for European options. Neither the BI nor the TRI was ever off by more than 0.009 or by more than 0.54% in option value. Figure 4.3 shows a comparison of the BI and TRI with approximately one-million nodes for American options. The difference in option values between the two lattice types never varies by more than 0.012 or by more than 0.14%. The pricing accuracy of the one-million node BI and TRI for both European and American options adds confidence to our use of the one-million node BI and TRI as benchmarks for American options.

Figure 4.4 compares the BI and TRI to the CRR Lattice with 239 nodes. Although a 239 node binomial lattice is not computationally expensive, a Quadruple lattice with the same number of time steps (twenty) contains one-million nodes. The BI dominates the TRI in average squared error with 0.05 versus 0.69, while the TRI dominates the BI in average squared percentage error with 24.58 versus 89.29. Notice that the pricing accuracy of the CRR Lattice varies widely with the absolute expected

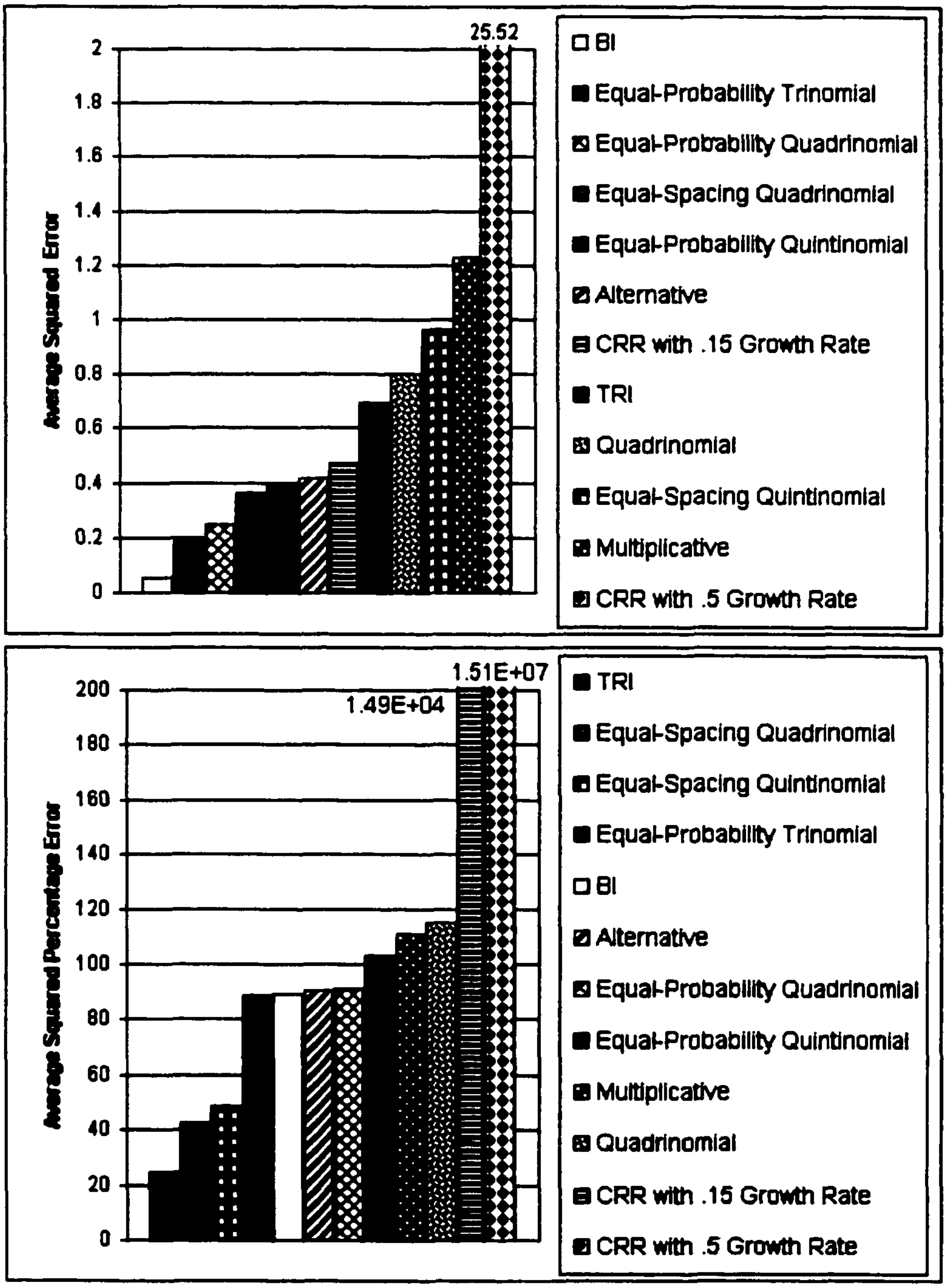
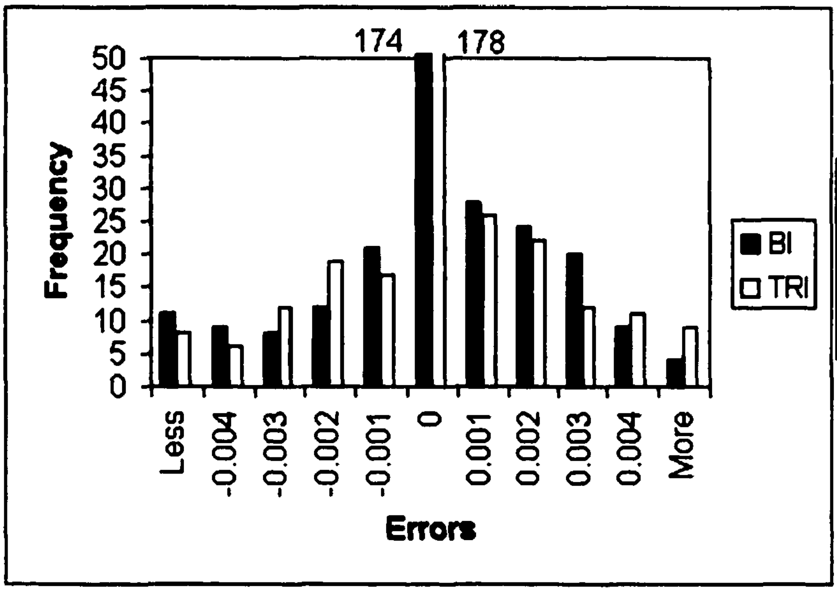
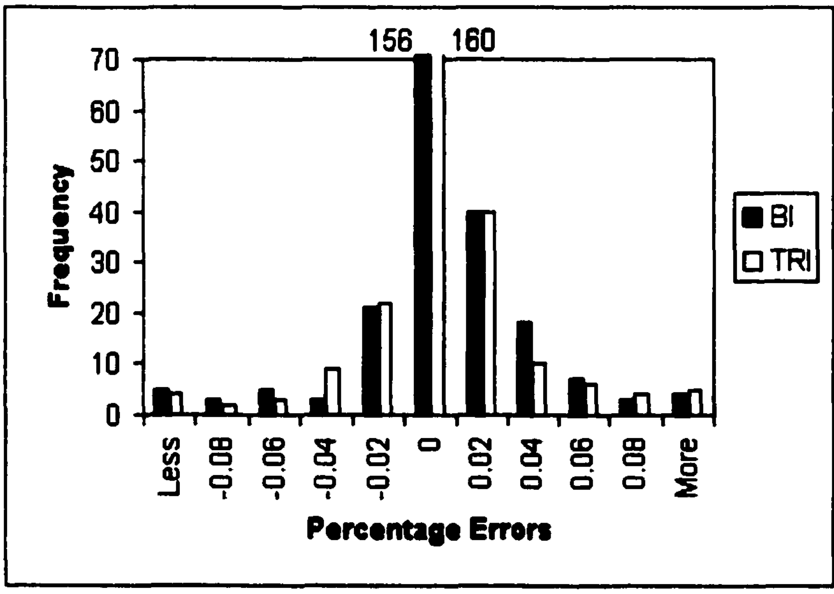


Figure 4.1: Lattice comparisons with 239 nodes.



	TRI	BI
AVE	0.00014	6E-05
AAE	0.00118	0.00116
ASE	4.2E-06	3.9E-06
MAXE	0.00815	0.00863
count	320	320



	TRI	BI
%AVE	0.00536	0.00453
%AAE	0.01832	0.01928
%ASE	0.00199	0.00221
%MAXE	0.50745	0.53662
count	270	270

Figure 4.2: BI and TRI with one-million nodes versus Black-Scholes.

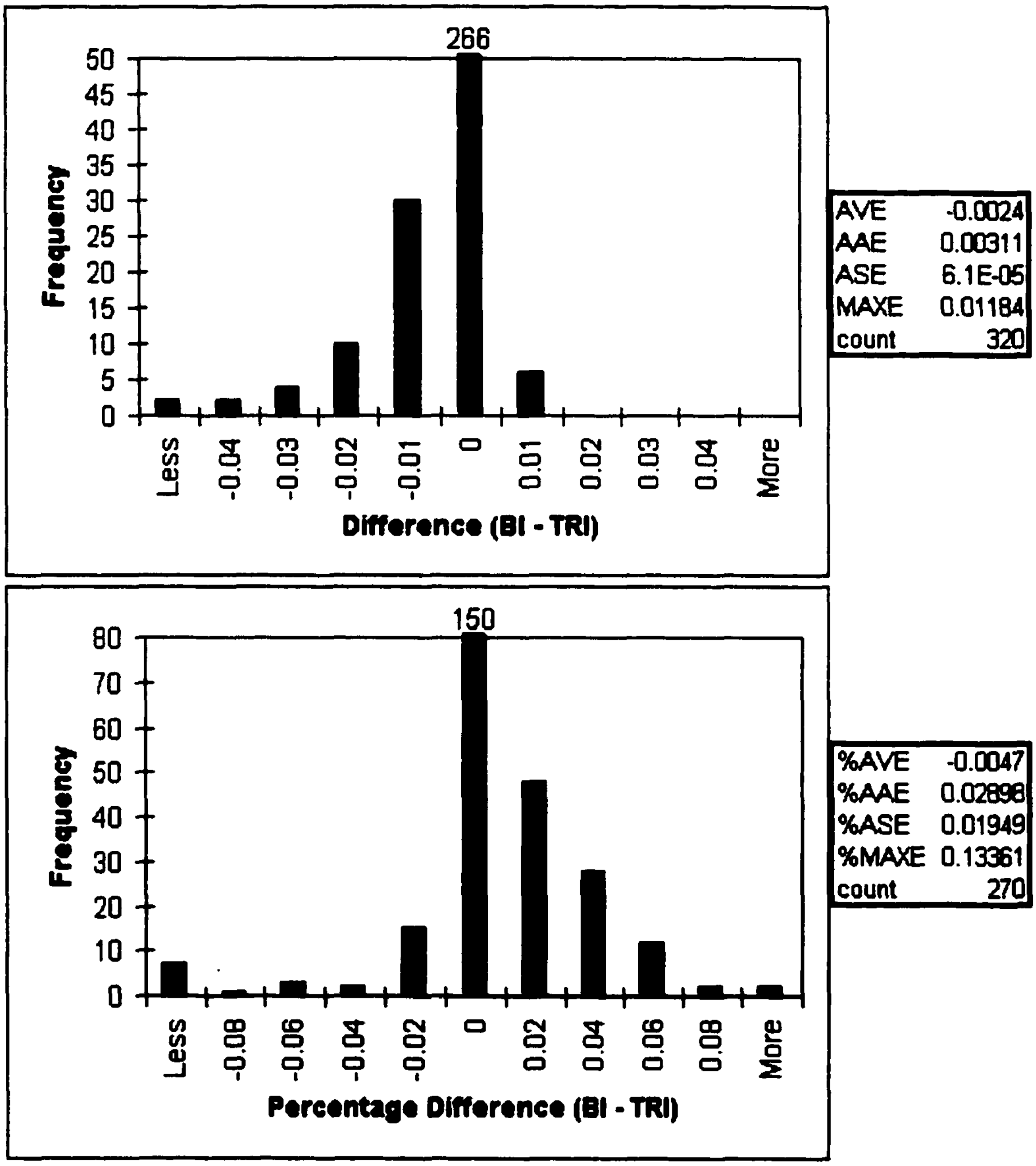


Figure 4.3: American BI and TRI differences with one-million nodes.

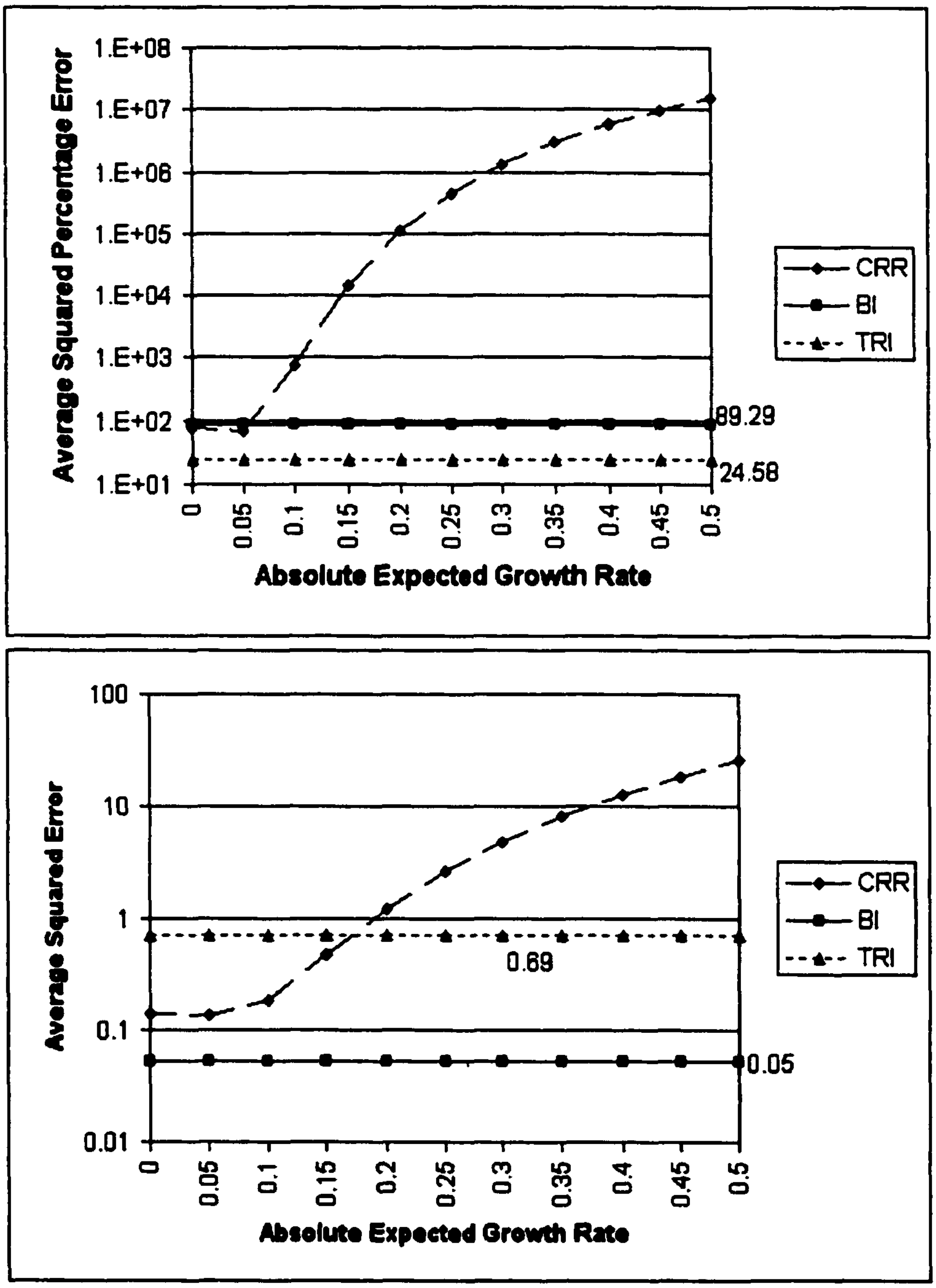
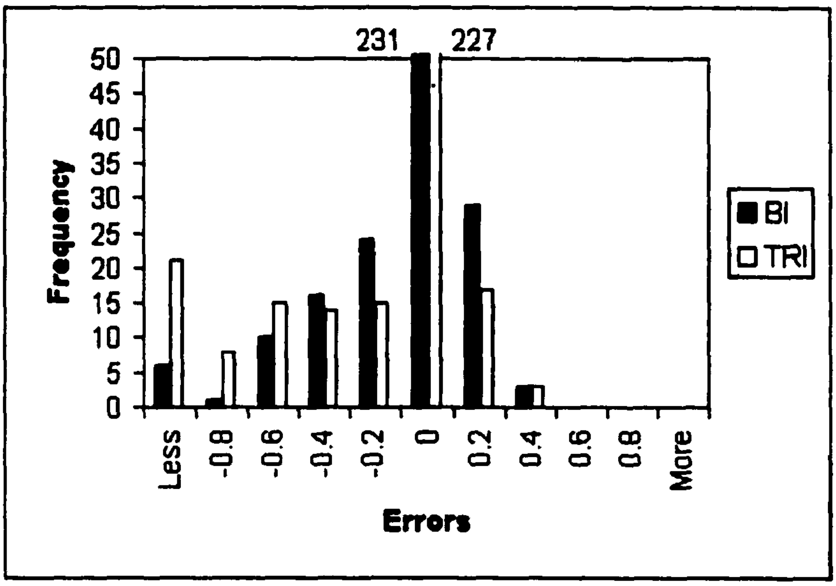


Figure 4.4: BI, TRI, and CRR Lattice with 239 nodes.

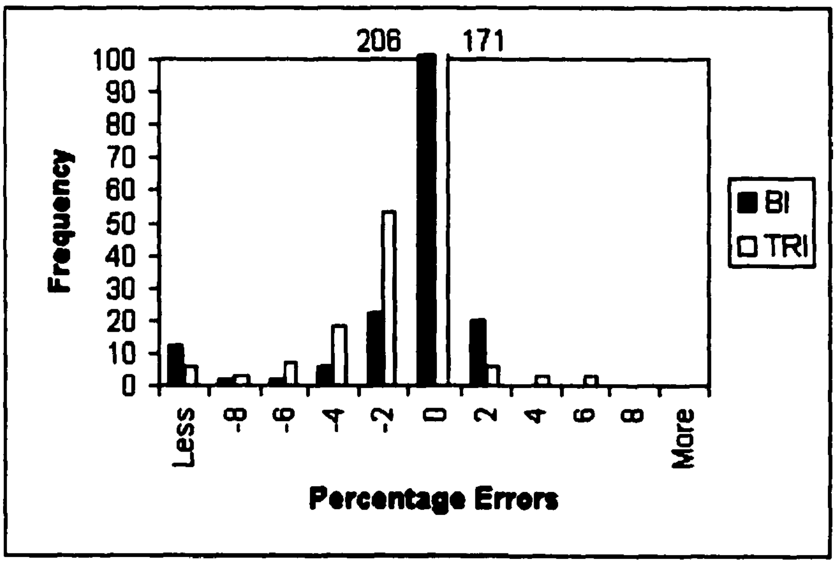
growth rate of the underlying variable. The CRR Lattice performs best at about a five percent absolute expected growth rate. As the absolute expected growth rate increases to 50%, the pricing accuracy of the CRR Lattice quickly diminishes resulting in an average squared error that is over 500 times higher than the BI and an average squared percentage error that is over 600,000 times higher than the TRI.

Figure 4.5 gives us a more detailed look at the accuracies of the BI and TRI with 239 nodes. On average, the BI and TRI undervalue options.

Figure 4.6 shows the rate of descent in average squared error and average squared percentage error of both the BI and TRI for the American options as we remove variables or increase the number of nodes in our lattices. The average squared error of the one-million node lattice was calculated with European options because of the lack of a more accurate benchmark for American options.



	BI	TRI
AVE	-0.0563	-0.235
AAE	0.10425	0.2666
ASE	0.05138	0.69432
MAXE	1.24974	7.48598
count	320	320



	BI	TRI
%AVE	-1.6797	-1.2568
%AAE	2.08723	1.68555
%ASE	89.2879	24.5802
%MAXE	100	50.2784
count	270	270

Figure 4.5: BI and TRI with 239 nodes.

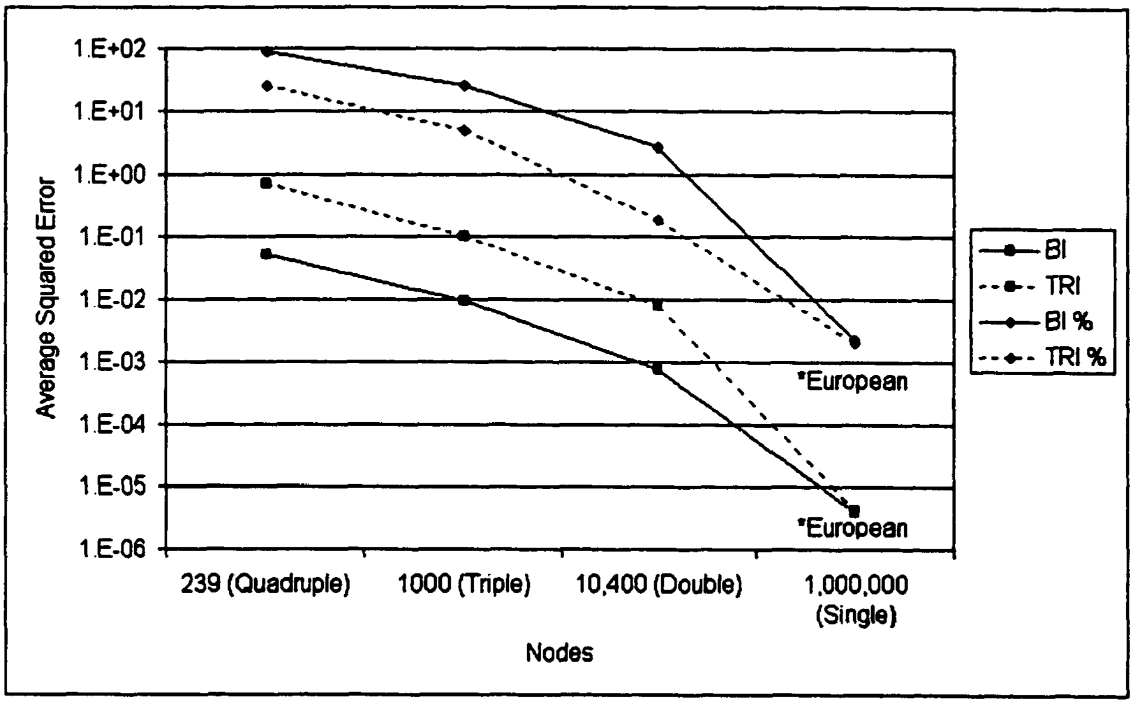


Figure 4.6: Convergence of BI and TRI in average squared error.

Chapter 5

Multiple Lattices

The CRR has been extended to *multiple lattices* (Luenberger 1998b; Kamrad and Ritchken 1991; He 1990; Madan, Milne, and Shefrin 1989; Boyle, Evnine, and Gibbs 1989; Boyle 1988), which are lattices that contain multiple variables. Other multiple lattice types have been constructed as well (Rubinstein 1994; Breen 1991; Omberg 1987; Hull 1993, pp. 428-9). Ekvall (1996) builds superior multiple binomial lattices by combining several BI lattices. He equates the risk-neutral probabilities and transforms the underlying variables so that they are independent. Finding the moves is complex and requires that a variance-covariance matrix be factorized and that the factors be inverted. For more information on financial numerical methods, see Trigeorgis (1996); Duffie (1996); and Hull (1993).

To make simple and intuitive multivariate approximations, we approximate each variable's distribution separately and then combine them. Thus, the sizes of the moves for the multiple lattices are the same as for the single lattices. The joint probabilities are found by matching the probabilities found separately and the *mixed moments* (moments containing more than one variable). We define the $(k_1, \ldots, k_l)$ moment as

$\mathrm{E}\left(\prod_{i=1}^{l} x_i^{k_i}\right)$. The wth moment has $\sum_{i=1}^{l} k_i = w$ and mixed moments have $k_i > 0$ and $k_j > 0$ for some $i \neq j$.

We assume that the logarithms of the underlying assets have a joint multivariate Normal distribution. This assumption simplifies our analysis so that we only need the means, variances, and correlations of the distributions to determine completely the joint distribution.

5.1 Fixed Move Method

In this section, we build multiple lattices by combining several BI lattices and then build other double lattices by combining the BI and trinomial lattices.

5.1.1 Double

We make a Double BI (Double) by putting together two BI lattices (see ch. 4). This lattice matches all of the third moments, is simple, and is unconditionally stable. Assume we have two underlying variables, A and B, with correlation $\rho_{A,B}$. Variable A has risk-neutral probability q_u^A of an up move and risk-neutral probability q_d^A of a down move. The lattice assumes that $\ln(A)$ moves up by u_A or down by d_A. We define the probabilities of the double lattice by double indexing, where the first index refers to A and the second to B; for example, q_{uu} is the risk-neutral probability of both A and B moving up. We can match the probabilities and the first mixed moment, E(AB), by solving the following set of equations:

$$\begin{aligned} q_u^A &= q_{uu} + q_{ud} \\ q_d^A &= q_{du} + q_{dd} \\ q_u^B &= q_{uu} + q_{du} \\ \mathrm{E}(AB) &= q_{uu}u_A u_B + q_{ud}u_A d_B + q_{du}d_A u_B + q_{dd}d_A d_B. \end{aligned}$$

The risk-neutral probabilities are

$$q_{uu} = q_{dd} = \frac{1 + \rho_{A,B}}{4}$$

and

$$q_{ud} = q_{du} = \frac{1 - \rho_{A,B}}{4}.$$

5.1.2 Other Multiple BI Lattices

Except for the Double lattice, Fixed-Move Multiple BI lattices are unstable for some correlation values. The Triple BI (Triple) is an example, although the results are simple for most correlation values. Assuming we have underlying variables, A, B, and C, the risk-neutral probabilities for the Triple are

$$\begin{aligned}
q_{uuu} = q_{ddd} &= \frac{1 + \rho_{A,B} + \rho_{B,C} + \rho_{A,C}}{8} \\
q_{udu} = q_{dud} &= \frac{1 - \rho_{A,B} - \rho_{B,C} + \rho_{A,C}}{8} \\
q_{uud} = q_{ddu} &= \frac{1 + \rho_{A,B} - \rho_{B,C} - \rho_{A,C}}{8} \\
q_{udd} = q_{duu} &= \frac{1 - \rho_{A,B} + \rho_{B,C} - \rho_{A,C}}{8}.
\end{aligned}$$

If any of the risk-neutral probabilities do not fall between 0 and 1, however, we use the Fixed Probability Method to build the lattice (see sec. 5.2).

An example of a Quadruple BI (Quadruple) risk-neutral probability is

$$q_{uuuu} = \frac{1 + \rho_{A,B} + \rho_{B,C} + \rho_{C,D} + \rho_{A,C} + \rho_{A,D} + \rho_{B,D} + M}{16},$$

where

$$M = \rho_{A,D}\rho_{B,C} + \rho_{A,C}\rho_{B,D} + \rho_{A,B}\rho_{C,D}.$$

In general, we find the Quadruple risk-neutral probabilities by following a pattern. Start with the number 1, and for each pair of variables that moves in the same direction, add their correlation coefficient. For each pair of variables that moves in the opposite direction, subtract their correlation coefficient. If there is an even number of up movements, add M; if there is an odd number, subtract M. Divide the whole by 16.

Table 5.1 compares the Multiple BI Lattices up to the Quintuple BI (Quintuple) by the number of moments matched, the order of the number of nodes per time step,

Table 5.1: Multiple BI Lattices

	Moments Matched	Order of Nodes Per Time Step	Time Steps in 1M Nodes	Mixed Moments Beyond Third
Double	7	$n^3/3$	143	None
Triple	13	$n^4/4$	43	None
Quadruple	23	$n^5/5$	20	E($ABCD$)
Quintuple	41	$n^6/6$	12	Many

the number of time steps in one-million nodes, and the mixed moments matched beyond the third moment.

5.1.3 Other Double Lattice Types

We are not limited to constructing multiple lattices from only BI lattices. It is difficult to pinpoint the best multiple lattice to use in a given situation but our test results of single lattices help. There are several important things in choosing an appropriate lattice: the region of the optimal exercise point; the correlation between the underlying variables; and the importance of each variable in the value of the option.

The single lattice test results tell us that to increase pricing accuracies for options with outlying exercise points, we should use the TRI;.for other options, we should use the BI. If one variable is significantly more important in accurately valuing the option than the other variable, we should consider using a double lattice that is a combination of a higher-order lattice and a lower-order lattice. Therefore, we construct double lattices out of a TRI and BI (Tri / Bi), two TRI lattices (Double Tri), two TRI lattices with $q_{uu} = q_{dd} = 0$ or $q_{ud} = q_{du} = 0$ (Correlated Tri), and an Equal-Probability Trinomial and BI (Eq-Tri / Bi). The risk-neutral probabilities and stable regions of

Table 5.2: Other Double Lattices

	Stable Region	$q_{uu} = q_{dd}$	$q_{ud} = q_{du}$	$q_{um}* = q_{dm}*$ $q_{mu} = q_{md}$	$q_{mm}*$
Tri / Bi	$\lvert\rho\rvert \leq \sqrt{3}/3$	$\frac{1+\sqrt{3}\rho}{12}$	$\frac{1-\sqrt{3}\rho}{12}$	$1/3$	
Double Tri*	$\lvert\rho\rvert \leq 1/2$	$\frac{(\rho+1)(\rho+1/2)}{18}$	$\frac{(\rho-1)(\rho-1/2)}{18}$	$(1-\rho^2)/9$	$(4+2\rho^2)/9$
Corr Tri*	$\rho > 1/2$	$\rho/6$	0	$(1-\rho)/6$	$(1+\rho)/3$
Corr Tri*	$\rho < -1/2$	0	$-\rho/6$	$(1+\rho)/6$	$(1-\rho)/3$
Eq-Tri / Bi	$\lvert\rho\rvert \leq \sqrt{6}/3$	$\frac{2+\sqrt{6}\rho}{12}$	$\frac{2-\sqrt{6}\rho}{12}$	$1/6$	

* Risk-neutral probabilities marked with a '*' are only valid for similarly marked lattices.

Table 5.3: Double Lattice Comparisons

	Moments Matched	Order of Nodes Per Time Step	Time Steps in 1M Nodes	Mixed Moments Beyond Third
Tri / Bi	10	$2n^3/3$	114	None
Eq-Tri / Bi	9	$2n^3/3$	114	None
Double Tri	14	$4n^3/3$	90	E(A^2B^2)
Correlated Tri	13	n^3	99	None

these four double lattice types, which each match at least all the third moments, are listed in table 5.2.

For each of these double lattice types, table 5.3 shows the same comparisons that are in table 5.1. Table 5.4 shows the asymptotic comparisons of all the double lattice types. Table 5.5 shows which double lattices are likely to be the most accurate when the importance of the variables is similar or unequal, when the absolute correlation is high or low, and when the optimal exercise points are at the edges of the variables' distributions or not. Note that the Double is always the simplest, most often the most accurate, and handles most problems well.

Table 5.4: Asymptotic Comparisons

	Time Steps	End Nodes	Average Spacing	Total Range
Double	$\sqrt[3]{3z}$	$\sqrt[3]{9z^2}$	$\sqrt[6]{64/3z}$	$2\sqrt[6]{3z}$
Tri / Bi	$\sqrt[3]{3z/2}$	$\sqrt[3]{18z^2}$	$\sqrt[6]{18/z}, \sqrt[6]{128/3z}$	$2\sqrt[6]{81z/2}, 2\sqrt[6]{3z/2}$
Eq-Tri / Bi	$\sqrt[3]{3z/2}$	$\sqrt[3]{18z^2}$	$\sqrt[6]{9/4z}, \sqrt[6]{128/3z}$	$2\sqrt[6]{81z/16}, 2\sqrt[6]{3z/2}$
Double Tri	$\sqrt[3]{3z/4}$	$\sqrt[3]{36z^2}$	$\sqrt[6]{36/z}$	$2\sqrt[6]{81z/4}$
Correlated Tri	$\sqrt[3]{z}$	$\sqrt[3]{27z^2}$	$\sqrt[6]{27/z}$	$3\sqrt[6]{z}$

Note: z is the number of nodes in the lattice.

Table 5.5: Double Lattice Regions

Importance Weighting	Absolute Correlation	None at the Edge	One at the Edge	Both at the Edge
Similar	Low	Double	Tri / Bi	Double Tri
Similar	High	Double	Correlated Tri	Correlated Tri
Unequal	Low	Eq-Tri / Bi	Tri / Bi	Double Tri
Unequal	High	Double	Correlated Tri	Correlated Tri

5.2 Fixed Probability Method

The Fixed Probability Method is more complex than the Fixed Move Method, but is stable for any number of variables and appears to be more consistently accurate.

5.2.1 Ekvall's Transformation

The transformation we outline here is derived from Ekvall (1996). Begin with N observables that follow a process:

$$\mathrm{d}\ln(Z_k) = \nu_{Z_{kQ}}\mathrm{d}t + \sigma_{Z_k}\mathrm{d}\hat{\mathrm{B}}_k \text{ for } k = 1, \dots, N;$$

where $\hat{\mathrm{B}}_k$ is a risk-neutral Brownian motion, $\nu_{Z_{kQ}}$ is the risk-neutral expected growth rate of observable Z_k, and σ_{Z_k} is the volatility of observable Z_k. Let $\mathrm{d}\ln(\boldsymbol{Z})$ represent the vector of observables,

$$\mathrm{d}\ln(\boldsymbol{Z}) = [\mathrm{d}\ln(Z_1) \quad \mathrm{d}\ln(Z_2) \quad \dots \quad \mathrm{d}\ln(Z_N)]^{\mathrm{T}},$$

where T is transpose. Assume that the above processes have an N-variate Normal distribution, N_N:

$$\mathrm{d}\ln(\boldsymbol{Z}) \sim \mathrm{N}_N(\boldsymbol{\nu}_{Z_Q}\mathrm{d}t, \boldsymbol{\Omega}_Z\mathrm{d}t),$$

where ν_{Z_Q} is the vector of risk-neutral expected growth rates and Ω_Z is the variance-covariance matrix of the Normal Distribution,

$$\Omega_Z = \begin{bmatrix} \sigma^2_{Z_1} & \sigma_{Z_1}\sigma_{Z_2}\rho_{Z_1,Z_2} & \cdots & \cdots \\ \sigma_{Z_1}\sigma_{Z_2}\rho_{Z_1,Z_2} & \sigma^2_{Z_2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \sigma^2_{Z_N} \end{bmatrix}.$$

Our goal is to transform the observables, $\mathrm{d}\ln(\boldsymbol{Z})$, so that the variance-covariance matrix is the identity matrix, **I**. The variance-covariance matrix can be factorized into $\Omega_Z = \boldsymbol{K}\boldsymbol{K}^T$, where $\boldsymbol{K}$ is a lower triangular matrix (Ekvall 1996). The transformed variables are $\mathrm{d}\ln(\boldsymbol{Y}) = \boldsymbol{K}^{-1}\mathrm{d}\ln(\boldsymbol{Z})$. Then we see that the vector of risk-neutral expected growth rates of the transformed variables is $\nu_{Y_Q} = \boldsymbol{K}^{-1}\nu_{Z_Q}$ and that the transformed variance-covariance matrix is

$$\Omega_Y = \boldsymbol{K}^{-1}(\boldsymbol{K}\boldsymbol{K}^T)(\boldsymbol{K}^{-1})^T = \mathbf{I}.$$

5.2.2 Transformed Lattices

We can easily build multiple lattices out of combinations of the BI and trinomial lattices. These multiple lattices include new lattices like the Triple Tri and lattices that we built in the previous sections (e.g., Double, Double Tri, Bi/Tri, Triple). We first approximate the movements of the transformed variables. For example, a Multiple BI is approximated as

$$\Delta\ln(\boldsymbol{Y}) = \boldsymbol{K}^{-1}\nu_{Z_Q}\Delta t + \pm\sqrt{\Delta t};$$

where

$$\pm = [\pm_1 \quad \pm_2 \quad \dots \quad \pm_N]^{\mathrm{T}}$$

represents the vector of plus or minus signs that depend upon the possible up or down moves of the N transformed variables, $\Delta t = T/n$ is the length of the time step, T is the time length of the option, and n is the number of time steps. (For a trinomial lattice, $\pm$ represents the vector of plus signs, minus signs, or no sign that depend upon the possible up, down, or middle moves. And, the size of the changes is $\sqrt{3}$ larger than the BI.) We can see the movements of $\boldsymbol{Z}$ by multiplying through by $\boldsymbol{K}$:

$$\Delta \ln(\boldsymbol{Z}) = \boldsymbol{\nu}_{Z_Q} \Delta t + \boldsymbol{K} \pm \sqrt{\Delta t}.$$

For example, a Double lattice has probability of any move equal to 1/4. The moves of $\ln(Z_1)$ are

$$\Delta \ln(Z_1) = \nu_{Z_{1Q}} \Delta t \pm_1 \sigma_{Z_1} \sqrt{\Delta t}, \tag{5.1}$$

and the moves of $\ln(Z_2)$ are

$$\Delta \ln(Z_2) = \nu_{Z_{2Q}} \Delta t + \left[\pm_1 \rho_{Z_1,Z_2} \pm_2 \sqrt{1 - \rho^2_{Z_1,Z_2}} \right] \sigma_{Z_2} \sqrt{\Delta t}, \tag{5.2}$$

where $\pm_1$ is positive if Y_1 moves up and negative if Y_1 moves down. A Double Tri lattice has probabilities:

$$q_{uu} = q_{dd} = q_{ud} = q_{du} = 1/36$$
$$q_{mm} = 4/9$$
$$q_{mu} = q_{md} = q_{um} = q_{dm} = 1/9.$$

The moves are

$$\Delta \ln(Z_1) = \nu_{Z_{1Q}} \Delta t \pm_1 \sigma_{Z_1} \sqrt{3\Delta t}$$

and

$$\Delta \ln(Z_2) = \nu_{Z_{2Q}} \Delta t + \left[\pm_1 \rho_{Z_1,Z_2} \pm_2 \sqrt{1 - \rho^2_{Z_1,Z_2}}\right] \sigma_{Z_2} \sqrt{3\Delta t},$$

where $\pm_1$ is positive if Y_1 moves up, is negative if Y_1 moves down, and has no sign if Y_1 moves middle.

5.3 Factor Model Lattices

The relationship between observables may be determined to emanate from random quantities called *factors*. A factor model is constructed to show these relationships (see, e.g., Luenberger (1998a, pp. 198-207)). Suppose we have j mutually-independent factors, $f_1, \ldots, f_j$, that follow a process:

$$\mathrm{d} \ln(f_i) = \nu_{f_i} \mathrm{d}t + \sigma_{f_i} \mathrm{dB}_{f_i} \text{ for } i = 1, \ldots, j;$$

where B_{f_i} is a Brownian motion. Also suppose that there are N observables, $Z_1, \ldots, Z_N$, that follow a process:

$$\mathrm{d}\ln(Z_k) = \nu_{Z_k}\mathrm{d}t + \sigma_{Z_k}\mathrm{dB}_k = a_k\mathrm{d}t + \sum_{i=1}^{j} b_{k,i}\,\mathrm{d}\ln(f_i) \text{ for } k = 1, \ldots, N;$$

where $b_{k,i}$ represents the sensitivity of the kth observable to the ith factor and a_k is a constant. Notice that the factor model is consistent with our assumption that the returns of the observables have a multivariate Normal distribution. Let $\mathrm{d}\ln \boldsymbol{Z}$ represent the vector of observables,

$$\mathrm{d}\ln \boldsymbol{Z} = \boldsymbol{a}\mathrm{d}t + \boldsymbol{b}\mathrm{d}\ln \boldsymbol{f},$$

where $\mathrm{d}\ln \boldsymbol{f}$ is the j-dimensional vector of factors, $\boldsymbol{a}$ is the N-dimensional vector of constants, and $\boldsymbol{b}$ is the $N \times j$ matrix,

$$\boldsymbol{b} = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & \cdots \\ b_{2,1} & b_{2,2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & b_{N,j} \end{bmatrix}.$$

To use the factors for pricing, we must find the risk-neutral growth rates of the factors. Assume that f_1 is correlated with the Market portfolio (no more than one factor can be correlated with the Market portfolio since the factors are independent). The expected growth rate of f_1 can be transformed into the risk-neutral expected growth rate, $\nu_{f_{1Q}}$, using G–3(3.11). For all other factors, the risk-neutral expected growth rate is equal to the expected growth rate.

With the factor model in place, we can easily build multiple lattices out of combinations of the BI and trinomial lattices by approximating the movements of the factors. For example, a Multiple BI is approximated as

$$\Delta \ln \boldsymbol{f} = \boldsymbol{\nu}_{f_Q} \Delta t + \pm\sqrt{\Delta t};$$

where $\pm$ represents the vector of signs that depend upon the possible moves of the j factors. The movements of the observables are then

$$\Delta \ln \boldsymbol{Z} = (\boldsymbol{a} + \boldsymbol{b}\boldsymbol{\nu}_{f_Q})\Delta t + \boldsymbol{b}\pm\sqrt{\Delta t}.$$

For example, a Double lattice has probability of any move equal to 1/4. The moves of $\ln(Z_1)$ are

$$\Delta \ln(Z_1) = (a_1 + b_{1,1}\nu_{f_{1Q}} + b_{1,2}\nu_{f_{2Q}})\Delta t + (\pm_1 b_{1,1}\sigma_{f_1} \pm_2 b_{1,2}\sigma_{f_2})\sqrt{\Delta t},$$

and the moves of $\ln(Z_2)$ are

$$\Delta \ln(Z_2) = (a_2 + b_{2,1}\nu_{f_{1Q}} + b_{2,2}\nu_{f_{2Q}})\Delta t + (\pm_1 b_{2,1}\sigma_{f_1} \pm_2 b_{2,2}\sigma_{f_2})\sqrt{\Delta t},$$

where $\pm_1$ is positive if factor f_1 moves up and negative if f_1 moves down.

If the number of factors exceeds the number of observables by more than one or two, or if it is difficult to find the factors, use the Fixed Move Method or the Fixed Probability Method to build the lattice.

5.4 Application — Internet Advertising Space

A large software company wishes to explore the possibility of advertising on a hot website. The company's advertising experts predict that this website will attract software sales s at a rate of

$$\mathrm{d}\ln(s) = .25\mathrm{d}t + .5\mathrm{dB}_s,$$

with initial value of $s(0) = 3000$ per year and $\rho_{s,M} = .3$. These advertisements require administration costs c at a rate of

$$\mathrm{d}\ln(c) = .25\mathrm{dB}_c,$$

with initial value of $c(0) = \$3$ million per year, $\rho_{c,M} = .1$, and $\rho_{c,s} = .5$. The price of the software is \$1000. The company's financial experts predict that the Market portfolio follows the process:

$$\mathrm{d}\ln(M) = .14\mathrm{d}t + .3\mathrm{dB}_M;$$

and that the risk-free rate is constant at 0.05. Find the value to the company of a one-year lease of this advertising space. The space can be forfeited without cost but once forfeited, the space cannot be reclaimed.

Table 5.6: Costs and Sales Moves

	up,up	up,down	down,up	down,down
ln(c)	0.02489	0.02489	-.02511	-.02511
ln(s)	0.07013	-.01648	0.02013	-.06648

From G-3(3.11), the risk-neutral growth rate of c is

$$\begin{aligned}\nu_{c_Q} &= \nu_c - \beta_{c,M}\left(\nu_M + \frac{1}{2}\sigma_M^2 - r_f\right)\\ &= 0 - \frac{.1(.25)}{.3}(.14 + .045 - .05) = -.01125.\end{aligned}$$

The risk-neutral growth rate of s is found similarly as 0.1825. We break this continuous model into 100 time periods, use Ekvall's transformation, and then approximate with a Double lattice. From (5.1),

$$\begin{aligned}\Delta \ln(c) &= \nu_{c_Q}\Delta t \pm_1 \sigma_c\sqrt{\Delta t}\\ &= -.01125\left(\frac{1}{100}\right) \pm_1 .25\left(\frac{1}{10}\right),\end{aligned}$$

and from (5.2),

$$\begin{aligned}\Delta \ln(s) &= \nu_{s_Q}\Delta t + \left[\pm_1\rho_{c,s} \pm_2 \sqrt{1-\rho_{c,s}^2}\right]\sigma_s\sqrt{\Delta t}\\ &= .1825\left(\frac{1}{100}\right) + \left[\pm_1\frac{1}{2} \pm_2 \frac{\sqrt{3}}{2}\right].5\left(\frac{1}{10}\right).\end{aligned}$$

The moves of ln(c) and ln(s) are found in table 5.6.

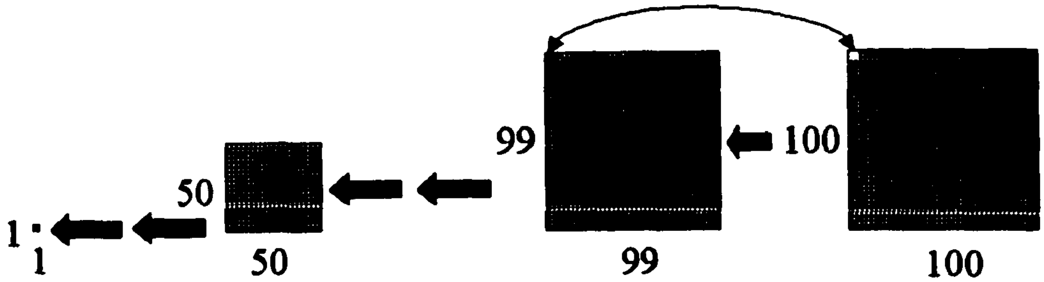


Figure 5.1: The problem structure.

The value of the lease at time t is:

$$V(t) = \begin{cases} \max[(1000s - c)/100, 0] & \text{for } t = 100 \\ \max\{(1000s - c)/100 + e^{-.0005}\hat{E}[V(t+1)], 0\} & \text{for } t = 0, \dots, 99; \end{cases}$$

where $\hat{E}[V(t+1)]$ is the risk-neutral expectation of the value of the lease in the next time period. We calculate the value of the lease with dynamic progromming as \$587,768. Figure 5.1 represents the possible costs and sales over time. It also shows the four possible moves that can be made from a node at time 99.

Bibliography

Amram, M. and N. Kulatilaka (1999). *Real Options : Managing Strategic Investment in an Uncertain World.* Financial Management Association Survey and Synthesis Series. Boston, MA: Harvard Business School Press.

Billingsley, P. (1995). *Probability and Measure* (3d ed.). Wiley Series in Probability and Mathematical Statistics. New York: John Wiley.

Boyle, P. P. (1988, March). A lattice framework for option pricing with two state variables. *Journal of Financial and Quantitative Analysis 23*(1), 1–12.

Boyle, P. P., J. Evnine, and S. Gibbs (1989). Numerical evaluation of multivariate contingent claims. *Review of Financial Studies 2*(2), 241–50.

Breen, R. (1991, June). The accelerated binomial option pricing model. *Journal of Financial and Quantitative Analysis 26*(2), 153–64.

Brennan, M. J. and E. S. Schwartz (1985, April). Evaluating natural resource investments. *Journal of Business 58*(2), 135–57.

Cox, J. C., S. A. Ross, and M. Rubinstein (1979, July). Option pricing: A simplified approach. *Journal of Financial Economics 7*, 229–63.

Dixit, A. K. and R. S. Pindyck (1994). *Investment Under Uncertainty.* Princeton, NJ: Princeton.

Duffie, D. (1988). *Securities Markets.* Economic Theory, Econometrics, and Mathematical Economics. San Diego, CA: Academic Press.

Duffie, D. (1996). *Dynamic Asset Pricing Theory* (2d ed.). Economic Theory, Econometrics, and Mathematical Economics. Princeton, NJ: Princeton.

Easton, S. A. (1996). A note on modified lattice approaches to option pricing. *The Journal of Futures Markets 16*(5), 585–94.

Ekvall, N. (1996, June 7). A lattice approach for pricing of multivariate contingent claims. *European Journal of Operational Research 91*(2), 214–28.

Gelman, A., J. B. Carlin, H. S. Stern, and D. B. Rubin (1995). *Bayesian Data Analysis* (1st ed.). London: Chapman & Hall.

Geske, R. and K. Shastri (1985, March). Valuation by approximation: A comparison of alternative option valuation techniques. *Journal of Financial and Quantitative Analysis 20*(1), 45–71.

He, H. (1990). Convergence from discrete- to continuous-time contingent claims prices. *Review of Financial Studies 3*(4), 523–46.

Huang, C. and R. H. Litzenberger (1988). *Foundations for Financial Economics.* Englewood Cliffs, NJ: Prentice-Hall.

Hull, J. C. (1993). *Options, Futures, and Other Derivative Securities* (2d ed.). Englewood Cliffs, NJ: Prentice Hall.

Jarrow, R. A. and A. Rudd (1983). *Option Pricing.* Homewood, IL: Richard D. Irwin.

Johnson, N. L. and S. Kotz (1969–1972). *Distributions in Statistics.* Wiley Series in Probability and Mathematical Statistics. New York: John Wiley. 4 vols.

Kamrad, B. and P. Ritchken (1991, December). Multinomial approximating models for options with k state variables. *Management Science 37*(12), 1640–52.

Kasanen, E. (1993, Autumn). Creating value by spawning investment opportunities. *Financial Management 22*(3), 251–8.

Kushner, H. J. and P. G. Dupuis (1992). *Numerical Methods for Stochastic Control Problems in Continuous Time.* Applications of Mathematics. New York: Springer-Verlag.

Law, A. M. and W. D. Kelton (n.d.). *Simulation Modeling and Analysis* (2d ed.). McGraw-Hill Series in Industrial Engineering and Management Science. New York: McGraw-Hill.

Luenberger, D. G. (1998a). *Investment Science.* New York: Oxford University Press.

Luenberger, D. G. (1998b). Products of trees for investment analysis. *Journal of Economic Dynamics and Control 22*, 1403–17.

Luenberger, D. G. (1999, October). Projection pricing. Submitted for publication.

Luenberger, D. G. (2000, March). A correlation pricing formula. In preparation.

Madan, D. B., F. Milne, and H. Shefrin (1989). The multinomial option pricing model and its Brownian and Poisson limits. *Review of Financial Studies 2*(2), 251–65.

McDonald, R. and D. Siegel (1984, March). Option pricing when the underlying asset earns a below-equilibrium rate of return: A note. *Journal of Finance 39*(1), 261–5.

McDonald, R. and D. Siegel (1985, June). Investment and the valuation of firms when there is an option to shut down. *International Economic Review 26*(2), 331–49.

Nawalkha, S. K. and D. R. Chambers (1995, August). The binomial model and risk neutrality: Some important details. *The Financial Review 30*(3), 605–15.

Oksendal, B. (1995). *Stochastic Differential Equations* (4th ed.). New York: Springer.

Omberg, E. (1987, June). A note on the convergence of binomial-pricing and compound-option models. *Journal of Finance 42*(2), 463–9.

Omberg, E. (1988, June). Efficient discrete time jump process models in option pricing. *Journal of Financial and Quantitative Analysis 23*(2), 161–74.

Paddock, J. L., D. R. Siegel, and J. L. Smith (1988, August). Option valuation of claims on real assets: The case of offshore petroleum leases. *The Quarterly Journal of Economics 103*(3), 479–508.

Rubinstein, M. (1994, November). Return to Oz. *Risk 7*(11), 67–71.

Smith, J. E. (1990, June). *Moment Methods for Decision Analysis*. Ph. D. thesis, Stanford University.

Tian, Y. (1993). A modified lattice approach to option pricing. *The Journal of Futures Markets 13*(5), 563–77.

Trigeorgis, L. (1991, September). A log-transformed binomial numerical analysis method for valuing complex multi-option investments. *Journal of Financial and Quantitative Analysis 26*(3), 309–26.

Trigeorgis, L. (1993a, March). The nature of option interactions and the valuation of investments with multiple real options. *Journal of Financial and Quantitative Analysis 28*(1), 1–20.

Trigeorgis, L. (1993b, Autumn). Rcal options and interactions with financial flexibility. *Financial Management 22*(3), 202–24.

Trigeorgis, L. (Ed.) (1995). *Real Options in Capital Investment: Models, Strategies, and Applications.* Westport, CT: Praeger Publishers.

Trigeorgis, L. (1996). *Real Options: Managerial Flexibility and Strategy in Resource Allocation.* Cambridge: MIT Press.